EE 362: Signals and Systems

The z-Transform and Discrete-Time Systems – Chapter 7

Lecture 21

Chapter Contents

- Z-Transform of a Discrete-Time Signal
- Properties of the z-transform
- Computation of the inverse z-transform
- Transfer Function Representation
- System Analysis Using the Transfer Function Representation

Consider the linear time-invariant discrete-time system given by the transfer function

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$$

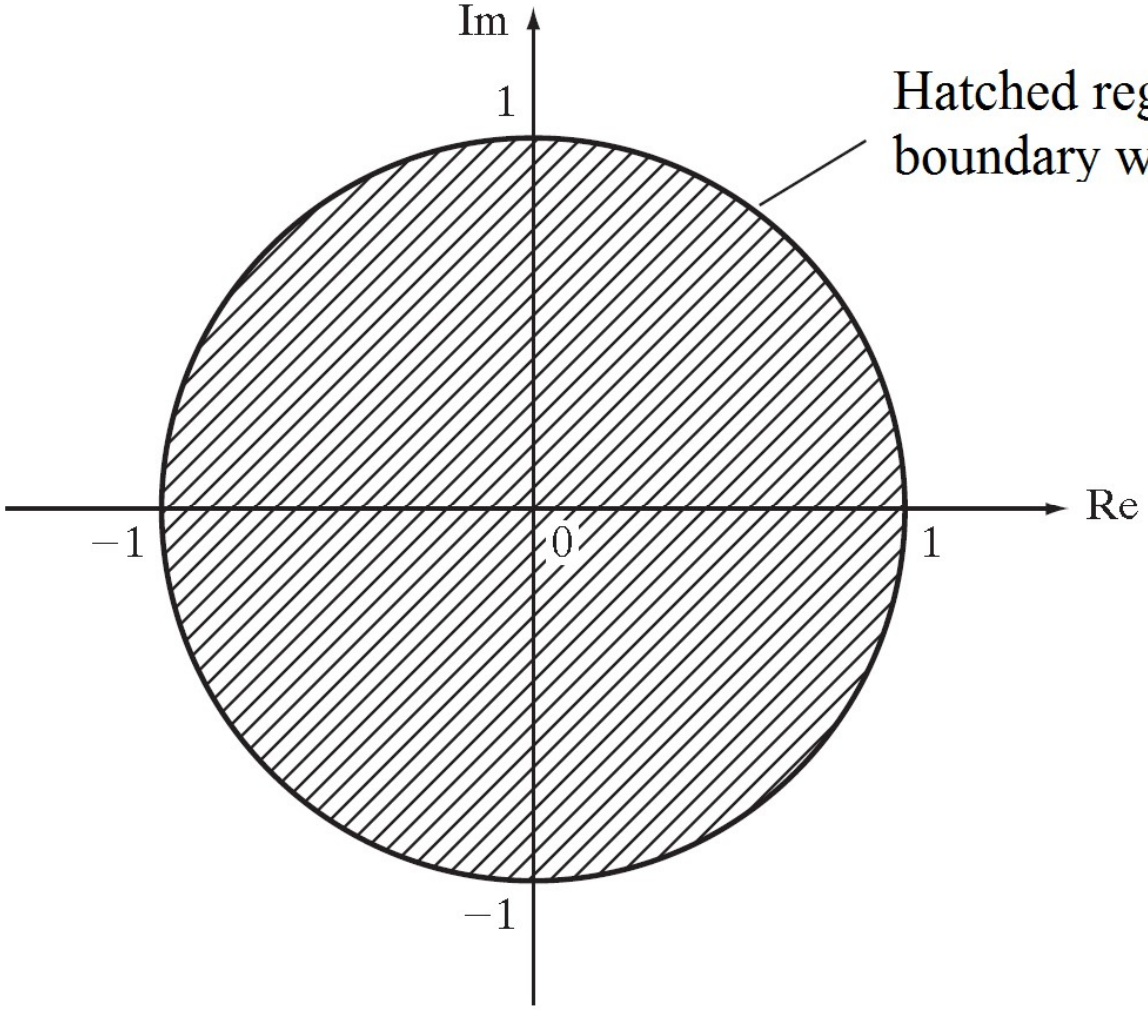
- The system is assumed to be causal, and thus $M \leq N$.
- It is also assumed that the polynomials $B(z)$ and $A(z)$ do not have any common factors.
- The transfer function $H(z)$ is the z-transform of the system's unit-pulse response $h[n]$.
- Time variation of $h[n]$ is directly determined by the poles of the system (roots of $A(z) = 0$).

- If $H(z)$ has a real pole p, then $h[n]$ contains a term of the form $c(p)^n$.
- If $H(z)$ has a complex pair of poles $a \pm jb$ with magnitude σ and angles $\pm\Omega$, then $h[n]$ contains a term of the form $c(\sigma)^n \cos(\Omega n + \theta)$.
- If $H(z)$ has repeated poles, then $h[n]$ will contain terms of the form $cn(n+1)\ldots(n+i)(p)^n$ and/or $cn(n+1)\ldots(n+i)(\sigma)^n \cos(\Omega n + \theta)$.
- It follows from the above points that $h[n]$ converges to zero as $n \to \infty$ if and only if

$$|p_i| < 1 \qquad \text{for } i = 1, 2, \ldots, N$$

where $p_1, p_2, \ldots, p_N$ are the poles of $H(z)$.

The condition $|p_i| < 1 \quad \text{for } i = 1, 2, \ldots, N$ is equivalent to requiring that all the poles be located in the **open unit disk** of the complex plane.



- A discrete-time system with transfer function $H(z)$ is said to be **stable** if its unit-pulse response $h[n]$ converges to zero as $n \to \infty$. In other words, *all system poles should lie in the open unit disk of the complex plane*.

- A discrete-time system with transfer function $H(z)$ is said to be **marginally stable** if its unit-pulse response $h[n]$ is bounded, that is

$$|h[n]| < c \quad \text{for all} \quad n$$

where c is a finite positive constant.

- In other words, a system is said to be **marginally stable**, if and only if $|p_i| \leq 1$ for all non-repeated poles of $H(z)$, and $|p_i| < 1$ for all repeated poles. This is equivalent to requiring that all poles lie in the open disk, except that non-repeated poles can be located on the **unit circle** ($|z| = 1$).
- A system is said to be **unstable** if the magnitude of $h[n]$ grows without bound as $n \to \infty$; that is,

$$|h[n]| \to \infty \quad \text{as } n \to \infty$$

- This is equivalent to having one or more poles located outside the closed disk or having one or more repeated poles located on the unit circle.

- Therefore, the stability boundary of the discrete-time case is the unit circle of the complex plane. What about continuous-time systems?
- The convergence of $h[n]$ to zero turns out to be equivalent (when $H(z)$ is rational) to absolute summability of $h[n]$; that is,

$$\sum_{n=0}^{\infty} |h[n]| < \infty$$

- Thus, stability is equivalent to absolute summability of the unit-pulse response $h[n]$.
- Convergence of $h[n]$ to zero is equivalent to **bounded-input bounded-output** (**BIBO**) stability.

- BIBO implies that the output $y[n]$ is a bounded signal whenever the input $x[n]$ is bounded (assuming zero initial conditions).
- In mathematical terms, BIBO stability means that whenever $|x[n]| \leq c_1$ for all n and for some finite positive constant c_1, then $|y[n]| \leq c_2$ for all n and for some finite positive constant c_2, where $y[n]$ is the response to $x[n]$ with zero initial conditions.
- Hence, stability of a system (i.e., convergence of $h[n]$ to zero) is equivalent to BIBO stability.
- Examples are provided separately.

Response to a Sinusoidal Input:

Consider the linear time-invariant discrete-time system given by the transfer function

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$$

Assume that the system is stable. Let the system input is given by,

$$x[n] = C\cos(\Omega_0 n), n = 0, 1, 2, \ldots$$

If we assume initial conditions, $Y(z)$ will be (details in the book)

$$Y(z) = \frac{z\eta(z)}{A(z)} + \frac{(C/2)H(e^{j\Omega_0})z}{z - e^{j\Omega_0}} + \frac{(C/2)\overline{H(e^{j\Omega_0})}z}{z - e^{-j\Omega_0}}$$

- Let $y_{tr}[n]$ denote the inverse of z-transform of the first term. Since the system is stable, the roots of $A(z) = 0$ are within the open disk of the complex plane, and thus $y_{tr}[n]$ converges to zero as $n \to \infty$. Hence, $y_{tr}[n]$ is the **transient part** of the output response resulting from the sinusoidal input $C\cos(\Omega_0 n)$.
- Let $y_{ss}[n]$ denote the inverse of z-transform of the second and third terms. By using the trigonometric identity, we can write,

$$y_{ss}[n] = C|H(e^{j\Omega_0})|\cos[\Omega_0 n + \angle H(e^{j\Omega_0})], \quad n = 0, 1, 2, \ldots$$

$$y_{ss}[n] = C|H(e^{j\Omega_0})| \cos[\Omega_0 n + \angle H(e^{j\Omega_0})], \quad n = 0, 1, 2, \ldots$$

- The response $y_{ss}[n]$ clearly does not converge to zero as $n \to \infty$, and thus it is the **steady-state part** of the output response resulting from the input $C\cos(\Omega_0 n)$.
- Then, the steady-state response resulting from a sinusoidal input is also a sinusoid with the same frequency, but the magnitude is scaled by $|H(e^{j\Omega_0})|$ and the phase is shifted by $\angle H(e^{j\Omega_0})$.
- This is similar to the results of Chapter 5.