EE 362: Signals and Systems

The z-Transform and Discrete-Time Systems – Chapter 7

Lecture 20

Chapter Contents

- Z-Transform of a Discrete-Time Signal
- Properties of the z-transform
- Computation of the inverse z-transform
- Transfer Function Representation
- System Analysis Using the Transfer Function Representation

Transfer Function Representation

Consider a causal linear time-invariant discrete-time system that can be represented by an input/output **difference** equation.

First-Order Case: $y[n] + ay[n-1] = bx[n]$

where a and b are real numbers, $y[n]$ is the output, and $x[n]$ is the input.

Second-Order Case:

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1]$$

Nth-Order Case:

$$y[n] + \sum_{i=1}^{N} a_i y[n-i] = \sum_{i=1}^{M} b_i x[n-i]$$

First-Order Case:

$$y[n] + ay[n-1] = bx[n]$$

Taking the z-transform gives,

$$Y(z) + a[z^{-1}Y(z) + y[-1]] = bX(z)$$

Solving for $Y(z)$

$$Y(z) = -\frac{ay[-1]}{1 + az^{-1}} + \frac{b}{1 + az^{-1}}X(z)$$

Or

$$Y(z) = -\frac{ay[-1]z}{z + a} + \frac{bz}{z + a}X(z)$$

If initial conditions are zero,

$$Y(z) = \frac{bz}{z + a}X(z) \quad \Rightarrow \quad H(z) = \frac{bz}{z + a}$$

Example: Suppose that $a \neq -1$ and $x[n] = u[n]$

Then $X(z) = z/(z-1)$.

$$Y(z) = -\frac{ay[-1]z}{z+a} + \frac{bz}{z+a}\left(\frac{z}{z-1}\right) = -\frac{ay[-1]z}{z+a} + \frac{bz^2}{(z+a)(z-1)}$$

Expanding the second term

$$\frac{bz^2}{(z+a)(z-1)}\frac{1}{z} = \frac{ab/(a+1)}{z+a} + \frac{b/(a+1)}{z-1}$$

$$y[n] = -ay[-1](-a)^n + \frac{b}{a+1}[a(-a)^n + (1)^n]$$

$$= -ay[-1](-a)^n + \frac{b}{a+1}[-(-a)^{n+1} + 1], \quad n = 0, 1, 2, \ldots$$

If $y[-1] = 0$, $y[n] = \frac{b}{a+1}[-(-a)n + 1], \quad n = 0, 1, 2, \ldots$

Second-Order Case:

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1]$$

Taking z-transform gives,

$$Y(z) + a_1[z^{-1}Y(z) + y[-1]] + a_2[z^{-2}Y(z) + z^{-1}y[-1] + y[-2]]$$

$$= b_0 X(z) + b_1 z^{-1} X(z)$$

Solving for $Y(z)$,

$$Y(z) = \frac{-a_2 y[-2] - a_1 y[-1] - a_2 y[-1] z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} + \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} X(z)$$

$$Y(z) = \frac{-(a_1 y[-1] + a_2 y[-2]) z^2 - a_2 y[-1] z}{z^2 + a_1 z + a_2} + \frac{b_0 z^2 + b_1 z}{z^2 + a_1 z + a_2} X(z)$$

If $y[-1] = y[-2] = 0$

$$Y(z) = \frac{b_0 z^2 + b_1 z}{z^2 + a_1 z + a_2} X(z) \quad \Rightarrow \quad H(z) = \frac{b_0 z^2 + b_1 z}{z^2 + a_1 z + a_2}$$

Example:

$$y[n] + 1.5y[n-1] + 0.5y[n-2] = x[n] - x[n-1]$$

Then,

$$H(z) = \frac{z^2 - z}{z^2 + 1.5z + 0.5}$$

The goal is to compute the output response when $y[-1] = 2$, $y[-2] = 1$, and $x[n] = u[n]$.

Using the analysis of the second-order case, we get,

$$Y(z) = \frac{-[(1.5)(2) + (0.5)(1)]z^2 - (0.5)(2)z}{z^2 + 1.5z + 0.5} + \frac{z^2 - z}{z^2 + 1.5z + 0.5}\left(\frac{z}{z-1}\right)$$

$$= \frac{-3.5z^2 - z}{z^2 + 1.5z + 0.5} + \frac{z^2}{z^2 + 1.5z + 0.5}$$

$$= \frac{-2.5z^2 - z}{z^2 + 1.5z + 0.5}$$

$$= \frac{0.5z}{z + 0.5} - \frac{3z}{z + 1}$$

Taking the inverse z-transform,

$$y[n] = 0.5(-0.5)^n - 3(-1)^n, \quad n = 0, 1, 2, \ldots$$

Nth-Order Case:

$$y[n] + \sum_{i=1}^{N} a_i y[n-i] = \sum_{i=1}^{M} b_i x[n-i]$$

Taking the z-transform of both sides and multiplying by z^2/z^2 yield,

$$Y(z) = \frac{C(z)}{A(z)} + \frac{B(z)}{A(z)} X(z)$$

where

$$B(z) = b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}$$

$$A(z) = z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N$$

If all initial conditions are zero

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N}$$

Suppose the system is represented by input/output convolution relationship.

$$y[n] = h[n] * x[n] = \sum_{i=0}^{n} h[i]x[n-i], \quad n = 0, 1, 2, \ldots$$

where the input is zero for $n < 0$ and all initial conditions are zero. Then,

$$Y(z) = H(z)X(z)$$

Where,

$$h[n] \leftrightarrow H(z)$$

- A discrete-time system is said to be finite dimensional if the transfer function $H(z)$ is a rational function of z.
- A discrete-time system is finite dimensional if and only if it can be described by an input/output difference equation.
- If a given discrete-time system is finite-dimensional, so that $H(z)$ is rational, the poles and zeros of the system are defined to be the poles and zeros of $H(z)$.

Example: Compute the transform of the following linear time-invariant discrete-time system

$$h[n] = 3(2^{-n}) \cos\left(\frac{\pi n}{6} + \frac{\pi}{12}\right), \quad n = 0, 1, 2, \ldots$$

Use $\cos(a + b) = (\cos a)(\cos b) - (\sin a)(\sin b)$

$$h[n] = 3(2^{-n})\left[\cos\left(\frac{\pi n}{6}\right)\cos\left(\frac{\pi}{12}\right) - \sin\left(\frac{\pi n}{6}\right)\sin\left(\frac{\pi}{12}\right)\right], \quad n \geq 0$$

$$= 2.898\left(\frac{1}{2}\right)^n \cos\left(\frac{\pi n}{6}\right) - 0.776\left(\frac{1}{2}\right)^n \sin\left(\frac{\pi n}{6}\right), \quad n \geq 0$$

$$H(z) = 2.898\frac{z^2 - [0.5\cos(\pi/6)]z}{z^2 - [\cos(\pi/6)]z + 0.25} - 0.776\frac{[0.5\sin(\pi/6)]z}{z^2 - [\cos(\pi/6)]z + 0.25}$$

$$= \frac{2.898z^2 - 1.449z}{z^2 - 0.866z + 0.25}$$

Example: Compute the Step Response of the previous example. The step response is the output resulting from the step input $x[n] = u[n]$ with zero initial conditions. Since, $X(z) = z/(z-1)$, then using the results of the previous example

$$Y(z) = H(z)X(z) \Rightarrow Y(z) = \frac{2.898z^3 - 1.449z^2}{(z-1)(z^2 - 0.866z + 0.25)}$$

Expanding $Y(z)$,

$$\frac{Y(z)}{z} = \frac{cz + d}{z^2 - 0.866z + 0.25} + \frac{c_3}{z-1}$$

Inverse z-transform (details in the book):

$$y[n] = -0.875\left(\frac{1}{2}\right)^n \cos\frac{\pi n}{6} + 2.26\left(\frac{1}{2}\right)^n \sin\frac{\pi n}{6} + 3.773, \quad n = 0, 1, 2, \ldots$$

Transfer Function of Interconnections:

The transfer function of a linear time-invariant discrete-time system can be computed directly from a **signal-flow diagram** of the system.

Interconnection of Unit-Delay Elements:

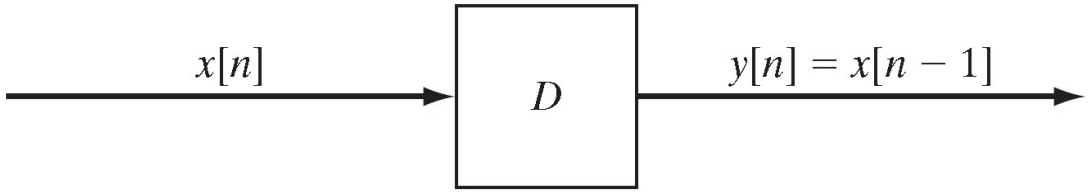
The unit-delay element is a system whose input/output relationship is given by

$$y[n] = x[n-1]$$

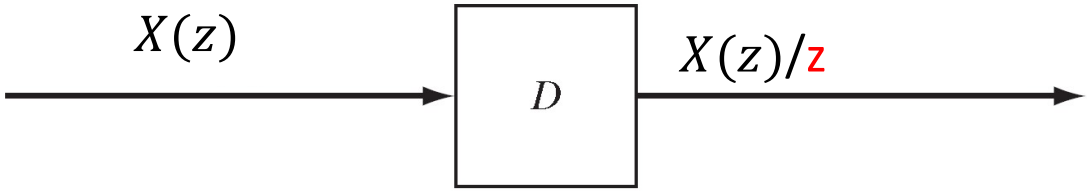
Taking the z-transform with zero initial conditions gives

$$Y(z) = z^{-1}X(z)$$

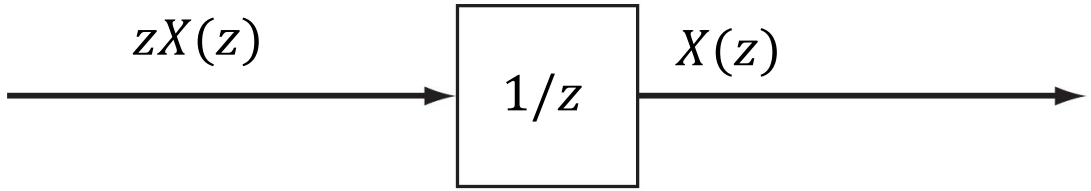
The unit-delay can be represented by the ‘D’ box as shown below,



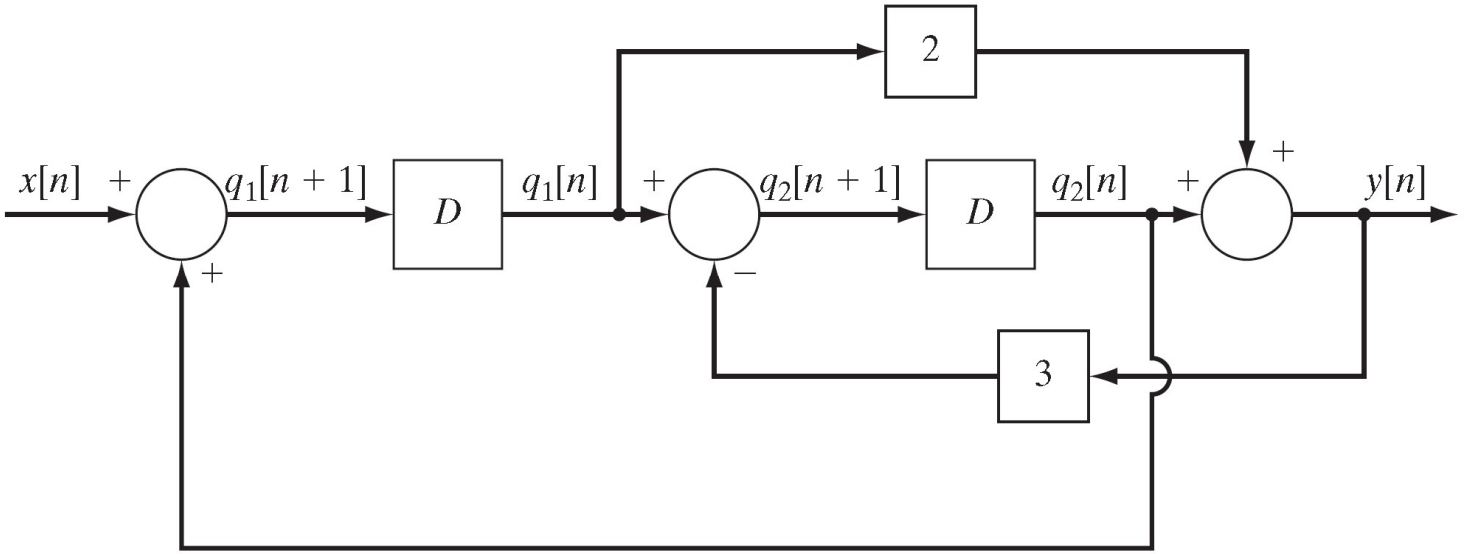
Taking the z-transform

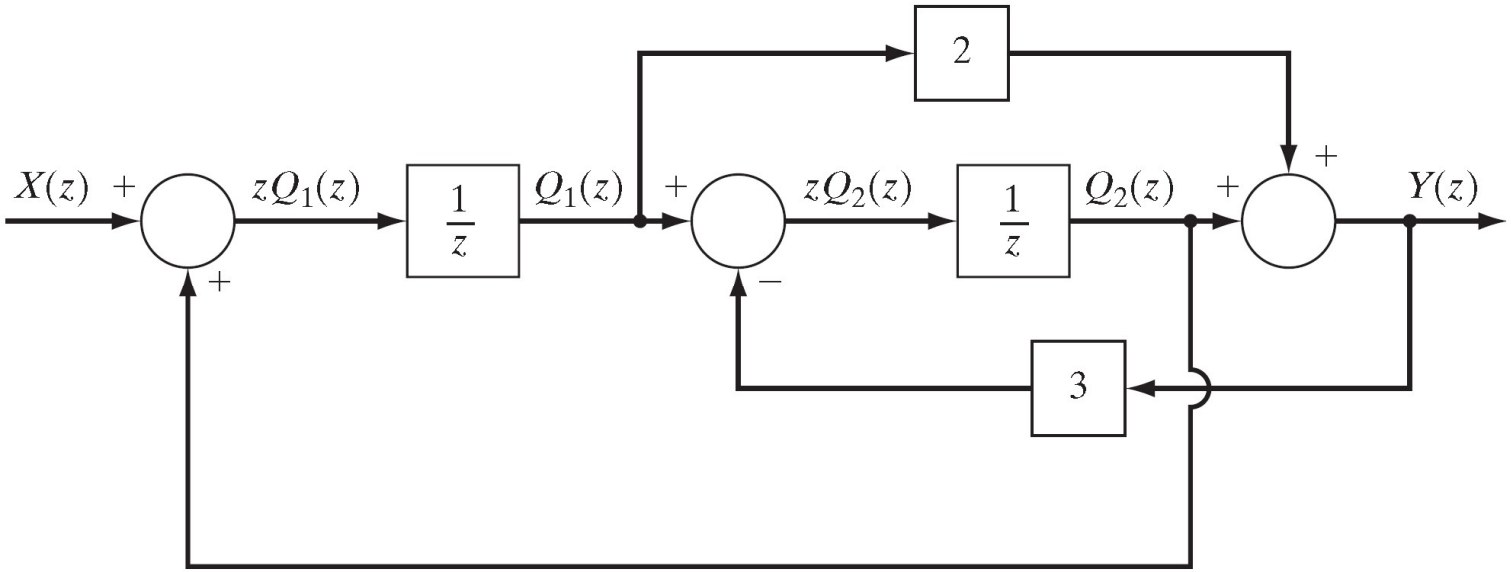


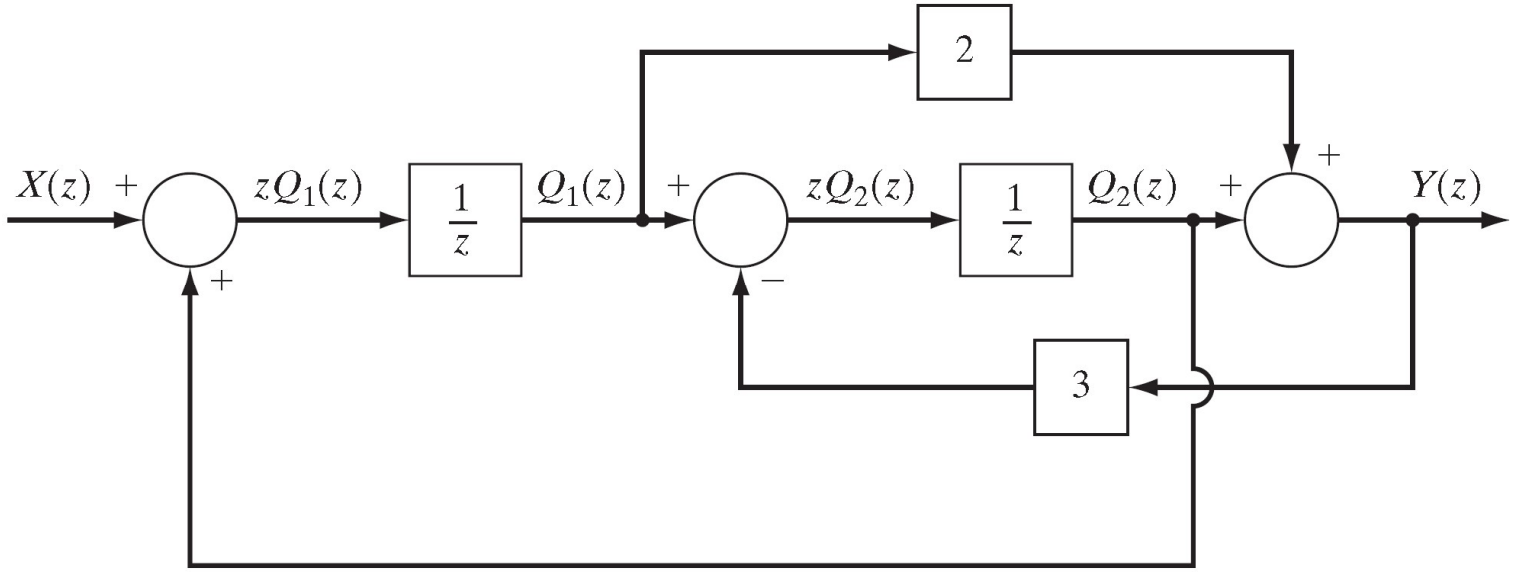
Or



Example:







$$zQ_1(z) = Q_2(z) + X(z)$$

$$zQ_2(z) = Q_1(z) - 3Y(z)$$

$$Y(z) = 2Q_1(z) + Q_2(z)$$

$$zQ_2(z) = z^{-1}Q_2(z) + z^{-1}X(z) - 3Y(z)$$

$$Y(z) = 2z^{-1}Q_2(z) + 2z^{-1}X(z) + Q_2(z)$$

$$Y(z) = \frac{2z^{-1} + 1}{z - z^{-1}}[z^{-1}X(z) - 3Y(z)] + 2z^{-1}X(z)$$

$$\left[1 + \frac{3(2z^{-1} + 1)}{z - z^{-1}}\right]Y(z) = \frac{z^{-1}(2z^{-1} + 1)}{z - z^{-1}}X(z) + 2z^{-1}X(z)$$

$$\frac{z + 5z^{-1} + 3}{z - z^{-1}}Y(z) = \left[\frac{z^{-1}(2z^{-1} + 1)}{z - z^{-1}} + 2z^{-1}\right]X(z) = \frac{z^{-1} + 2}{z - z^{-1}}X(z)$$

$$Y(z) = \frac{z^{-1} + 2}{z + 5z^{-1} + 3}X(z) = \frac{2z + 1}{z^2 + 3z + 5}X(z)$$

$$H(z) = \frac{2z + 1}{z^2 + 3z + 5}$$

Transfer Function of Basic Interconnections:

