EE 362: Signals and Systems

The z-Transform and Discrete-Time Systems – Chapter 7

Lecture 19

Chapter Contents

- Z-Transform of a Discrete-Time Signal
- Properties of the z-transform
- Computation of the inverse z-transform
- Transfer Function Representation
- System Analysis Using the Transfer Function Representation

- If $X(z)$ is the z-transform of the discrete-time signal $x[n]$, we can compute the signal $x[n]$ by taking the inverse z-transform of $X(z)$ given by:

$$x[n] = \frac{1}{j2\pi} \int X(z) z^{n-1} \, dz$$

- When the transform $X(z)$ is a rational function of z, we can compute the z-transform without having to evaluate the integral of the above equation.
- When $X(z)$ is rational, $x[n]$ can be computed by expansion of $X(z)$ into a power series of z^{-1} or by expansion of $X(z)$ into partial fraction.

Inverse z-transform via long division: Let $X(z)$ be given in the rational form $X(z) = B(z)/A(z)$ with polynomials $B(z)$ and $A(z)$ written in descending powers of z.

To compute the inverse z-transform $x[n]$ for a finite range of values of n, we can expand $X(z)$ into a power series in z^{-1} by dividing $A(z)$ into $B(z)$ using long division.

Example: Find the inverse z-transform using long division

$$X(z) = \frac{z^2 - 1}{z^3 + 2z + 4}$$

Dividing $A(z)$ into $B(z)$ gives,

$$
\begin{array}{r}
z^{-1} + 0z^{-2} - 3z^{-3} - 4z^{-4} + \cdots \\
\hline
z^3 + 2z + 4 \,\big)\, z^2 - 1 \\
\underline{z^2 + 2 \quad + 4z^{-1}} \\
-3 \quad - 4z^{-1} \\
\underline{-3 \qquad\qquad - 6z^{-2} - 12z^{-3}} \\
-4z^{-1} + 6z^{-2} + 12z^{-3} \\
\underline{-4z^{-1} \qquad\qquad - \; 8z^{-3} - 16z^{-4}} \\
6z^{-2} + 20z^{-3} + 16z^{-4} \\
\vdots
\end{array}
$$

Thus $X(z) = z^{-1} - 3z^{-3} - 4z^{-4} \ldots$

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

$$x[0] = 0, \quad x[1] = 1, \quad x[2] = 0, \quad x[3] = -3, \quad x[4] = -4, \ldots$$

Example: Find the inverse z-transform of $X(z)$ using the long division

$$X(z)=\frac{z}{z-1}$$

Solution:

$$X(z)=\frac{z}{z-1}=1+z^{-1}+z^{-2}+z^{-3}+\ldots=\sum_{n=0}^{\infty}x[n]z^{-n}=x[0]+x[1]z^{-1}+x[2]z^{-2}+x[3]z^{-3}+\ldots$$

$$x[0]=1,x[1]=1,x[2]=1,x[3]=1,\ldots \qquad x[n]=\begin{cases}1, n\geq 0\\ 0, o.w\end{cases}=u[n]$$

Example: Find the inverse z-transform of $X(z)$ using the long division

$$X(z)=\frac{z^2+1}{z^2-1}$$

Solution:

$$X(z)=\frac{z^2+1}{z^2-1}=1+2z^{-2}+2z^{-4}+\ldots$$

$$x[0]=1,x[1]=0,x[2]=2,x[3]=0,x[4]=2,\ldots$$

Using MATLAB: Instead of the carrying out the long division by hand, we can use MATLAB to compute $x[n]$, applying either the symbolic manipulator "iztrans" or the command "filter". Consider

$$X(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_0}$$

To solve using symbolic manipulator, use "iztrans"

To solve for the inverse z-transform numerically for $n = 0$ to $n = q$, use the commands

```
num = [b_M b_M-1 … b_0];
den = [a_N a_N-1 … a_0];
x = filter(num,den,[1 zeros(1,q)]);
```

Example: Inverse z-transform using MATLAB.

$$X(z) = \frac{8z^3 + 2z^2 - 5z}{z^3 - 1.75z + .75}$$

Symbolically:

```
syms X x z
X = (8*z^3+2*z^2-5*z)/(z^3-1.75*z+.75);
x = iztrans(X)
```

```
x =
2*(1/2)^n+2*(-3/2)^n+4
```

Numerically for $n = 0$ to $n = 10$:

```
num = [8 2 -5 0];
den = [1 0 -1.75 .75];
x = filter(num,den,[1 zeros(1,9)])
```

```
8    2    9    -2.5    14.25    -11.125    26.8125    -30.1563    55.2656
```

Example: Find the inverse z-transform of $X(z)$ using MATLAB.

$$X(z) = \frac{z}{z-1}$$

```
syms  X  x  z
X= (z)/(z-1);
x = iztrans(X)
```

x = 1

```
num=[1 0];
den=[1 -1];
filter(num,den,[1 zeros(1,9)])
```

ans = 1 1 1 1 1 1 1 1 1 1

Example:

$$X(z) = \frac{z^2+1}{z^2-1}$$

```
num=[1 0 1];
den=[1 0 -1];
filter(num,den,[1 zeros(1,9)])
```

ans =

1 0 2 0 2 0 2 0 2 0

Inverse z-transform via partial fraction expansion:

- Using long division as previously described, we can compute the inverse z-transform $x[n]$ of $X(z)$ for any finite range on integer values of n.
- However, if an analytical (closed-form) expression for $x[n]$ is desired that is valid for all $n \geq 0$, it is necessary to use partial fraction expansion.
- Suppose that $X(z)$ is given in the rational form $X(z) = B(z)/A(z)$. If the degree of $B(z)$ is equal to $A(z)$, the direct partial fraction cannot be applied. However, dividing $A(z)$ into $B(z)$ yields

$$X(z) = x[0] + \frac{R(z)}{A(z)}$$

where $x[0]$ is the initial value of the signal $x[n]$ at time $n = 0$, and $R(z)$ is a polynomial in z whose degree is strictly less than that of $A(z)$. The rational function $R(z)/A(z)$ can then be expanded by partial fraction.

- There is another approach that avoids having to divide $A(z)$ into $B(z)$; namely, first expand

$$\frac{X(z)}{z} = \frac{B(z)}{zA(z)}$$

Then multiply the result by z to get $X(z)$

Distinct Poles:

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} + \cdots + \frac{c_N}{z - p_N}$$

where $$c_0 = \left[z \frac{X(z)}{z} \right]_{z=0} = X(0)$$

$$c_i = \left[(z - p_i) \frac{X(z)}{z} \right]_{z=p_i}, \quad i = 1, 2, \ldots, N$$

Multiplying by z we get,

$$X(z) = c_0 + \frac{c_1 z}{z - p_1} + \frac{c_2 z}{z - p_2} + \cdots + \frac{c_N z}{z - p_N}$$

Taking the inverse z-transform

$$x[n] = c_0 \delta[n] + c_1 p_1^n + c_2 p_2^n + \cdots + c_N p_N^n, \quad n = 0, 1, 2, \ldots$$

Example: Find the inverse z-transform of $X(z)$ using the partial fraction expansion:

$$X(z) = \frac{z}{z^2 - 1}$$

$$\frac{X(z)}{z} = \frac{1}{z^2 - 1} \qquad z^2 - 1 = 0 \Rightarrow z = \pm 1 \qquad p_1 = 1, p_2 = -1$$

$$c_0 = \left[z \frac{X(z)}{z} \right]_{z=0} = 0$$

$$c_1 = ((z - p_1)\frac{X(z)}{z})|_{z=p_1} = ((z-1)\frac{X(z)}{z})|_{z=1} = \frac{1}{2}$$

$$c_2 = ((z - p_2)\frac{X(z)}{z})|_{z=p_2} = ((z+1)\frac{X(z)}{z})|_{z=-1} = -\frac{1}{2}$$

$$x[n] = c_1(p_1)^n + c_2(p_2)^n = \frac{1}{2}(1)^n u[n] - \frac{1}{2}(-1)^n u[n]$$

Distinct Complex Poles:

Suppose that the pole $p_1 = a + jb$ where $b \neq 0$. Then there should a complex conjugate pole.

$$c_1 p_1^n + \overline{c}_1 \overline{p}_1^n$$

This term can be expressed as

$$2|c_1|\sigma^n \cos(\Omega n + \angle c_1)$$

where $\sigma = |p_1|$ & $\Omega = \angle p_1$

Then,

$$x[n] = c_0\delta[n] + 2|c_1|\sigma^n \cos(\Omega n + \angle c_1) + c_3 p_3^n + \cdots + c_n p_N^n, n = 0, 1, 2, \ldots$$

Example: Find the inverse z-transform of $X(z)$ using the partial fraction expansion: $X(z) = \frac{1}{z^2+1}$

$$\frac{X(z)}{z} = \frac{1}{z(z^2+1)} \qquad z(z^2+1) = 0 \Rightarrow z = \pm j, 0 \qquad p_0 = 0, p_1 = j, p_2 = -j$$

$$c_0 = (z\frac{X(z)}{z})|_{z=p_0} = (X(z))|_{z=0} = X(0) = 1$$

$$c_1 = ((z-p_1)\frac{X(z)}{z})|_{z=p_1} = ((z-j)\frac{X(z)}{z})|_{z=j} = \frac{1}{j(2j)} = -\frac{1}{2}$$

$$c_2 = ((z-p_2)\frac{X(z)}{z})|_{z=p_2} = ((z+j)\frac{X(z)}{z})|_{z=-j} = \frac{1}{-j(-2j)} = -\frac{1}{2}$$

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} = \frac{1}{z} + \frac{-\frac{1}{2}}{z-j} + \frac{-\frac{1}{2}}{z+j} \Rightarrow X(z) = 1 + -\frac{1}{2}\frac{z}{z-j} - \frac{1}{2}\frac{z}{z+j}$$

$$x[n] = c_0\delta[n] + c_1(p_1)^n + c_2(p_2)^n = \delta[n] - \frac{1}{2}(j)^n u[n] - \frac{1}{2}(-j)^n u[n]$$

Example: Suppose that

$$X(z) = \frac{z^3 + 1}{z^3 - z^2 - z - 2}$$

$$A(z) = z^3 - z^2 - z - 2$$

MATLAB roots

$$p_1 = -0.5 - j0.866$$
$$p_2 = -0.5 + j0.866$$
$$p_3 = 2$$

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z + 0.5 + j0.866} + \frac{\bar{c}_1}{z + 0.5 - j0.866} + \frac{c_3}{z - 2}$$

$$c_0 = X(0) = \frac{1}{-2} = -0.5$$

$$c_1 = \left[(z + 0.5 + j0.866) \frac{X(z)}{z} \right]_{z=-0.5-j0.866} = 0.429 + j0.0825$$

$$c_3 = \left[(z - 2) \frac{X(z)}{z} \right]_{z=2} = 0.643$$

$$x[n] = -0.5\delta[n] + c_1(-0.5 - j0.866)^n + \bar{c}_1(-0.5 + j0.866)^n + 0.643(2)^n, \quad n = 0, 1, 2, \ldots$$

OR

$$|p_1| = \sqrt{(0.5)^2 + (0.866)^2} = 1 \qquad |c_1| = \sqrt{(0.429)^2 + (0.0825)^2} = 0.437$$

$$\angle p_1 = \pi + \tan^{-1}\frac{0.866}{0.5} = \frac{4\pi}{3}\text{rad} \qquad \angle c_1 = \tan^{-1}\frac{0.0825}{0.429} = 10.89^\circ$$

$2|c_1|\sigma^n \cos(\Omega n + \angle c_1)$ where $\sigma = |p_1|$ $\Omega = \angle p_1$

$$x[n] = -0.5\delta[n] + 0.874\cos\left(\frac{4\pi}{3}n + 10.89^\circ\right) + 0.643(2)^n, \quad n = 0, 1, 2, \ldots$$

MATLAB (poles & residues):

```
num = [1 0 0 1];
den = [1 -1 -1 -2 0];
[r,p] = residue(num,den)
```

```
r =
   0.6429
   0.4286 - 0.0825i
   0.4286 + 0.0825i
  -0.5000

p =
   2.0000
  -0.5000 + 0.8660i
  -0.5000 - 0.8660i
        0
```

MATLAB for a range of 0 to 20:

```
num = [1 0 0 1];
den = [1 -1 -1 -2];
x = filter(num,den,[1 zeros(1,19)]);
```

Repeated Poles: suppose that p_1 is repeated r times and that other $N - r$ poles are distinct.

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z - p_1} + \frac{c_2}{(z - p_1)^2} + \cdots + \frac{c_r}{(z - p_1)^r} + \frac{c_{r+1}}{z - p_{r+1}} + \cdots + \frac{c_N}{z - p_N}$$

$c_0 = X(0)$, and $c_{r+1}, c_{r+2}, \ldots, c_n$ are computed as before. And,

$$c_r = \left[(z - p_1)^r \frac{X(z)}{z} \right]_{z=p_1}$$

$$c_{r-1} = \left[\frac{d}{dz} (z - p_1)^r \frac{X(z)}{z} \right]_{z=p_1}$$

$$c_{r-2} = \frac{1}{2!} \left[\frac{d^2}{dz^2} (z - p_1)^r \frac{X(z)}{z} \right]_{z=p_1}$$

$$\vdots$$

$$c_{r-i} = \frac{1}{i!} \left[\frac{d^i}{dz^i} (z - p_1)^r \frac{X(z)}{z} \right]_{z=p_1}$$

Then, multiplying both sides by z,

$$X(z) = c_0 + \frac{c_1 z}{z - p_1} + \frac{c_2 z}{(z - p_1)^2} + \cdots + \frac{c_r z}{(z - p_1)^r} + \frac{c_{r+1} z}{z - p_{r+1}}$$

$$+ \cdots + \frac{c_N z}{z - p_N}$$

We can compute the inverse z-transform of terms

$$\frac{c_i z}{(z - p_1)^i}$$

For $i = 2$ and $i_2 = 3$ by using Table 7.2.

$$c_2 n (p_1)^{n-1} u[n] \leftrightarrow \frac{c_2 z}{(z - p_1)^2} \qquad \frac{1}{2} c_3 n(n-1)(p_1)^{n-2} u[n] \leftrightarrow \frac{c_3 z}{(z - p_1)^3}$$

For $i = 4, 5, \ldots$ by repeatedly using the multiplication by n property of the z-transform.

$$\frac{c_i}{(i-1)!} n(n-1)\cdots(n-i+2)(p_1)^{n-i-1} u[n-i+2] \leftrightarrow \frac{c_i z}{(z - p_1)^i}, \quad i = 4, 5, \ldots$$

Example: Find the inverse z-transform of $X(z)$ using the partial fraction expansion:

$$X(z) = \frac{1}{(z+1)^2}$$

$$\frac{X(z)}{z} = \frac{1}{z(z+1)^2} \qquad z(z+1)^2 = 0 \Rightarrow \begin{cases} z = p_0 = 0 & \text{simple pole} \\ z = p_1 = -1 & \text{repeated pole} \end{cases}$$

$$\frac{X(z)}{z} = \frac{1}{z(z+1)^2} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{(z-p_1)^2} = \frac{c_0}{z} + \frac{c_1}{z+1} + \frac{c_2}{(z+1)^2}$$

$$c_0 = (z\frac{X(z)}{z})\,|_{z=p_0} = (X(z))\,|_{z=0} = X(0) = 1$$

$$c_2 = ((z-p_1)^2\,\frac{X(z)}{z})\,|_{z=p_1} = ((z+1)^2\,\frac{X(z)}{z})\,|_{z=-1} = -1$$

$$c_1 = \frac{d}{dz}((z-p_1)^2\,\frac{X(z)}{z})\,|_{z=p_1} = \frac{d}{dz}(\frac{1}{z})\,|_{z=-1} = -\frac{1}{z^2}\,|_{z=-1} = -1$$

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z+1} + \frac{c_2}{(z+1)^2} = \frac{1}{z} + \frac{-1}{z+1} + \frac{-1}{(z+1)^2}$$

$$x[n] = \delta[n] - (-1)^n u[n] + n(-1)^{n-1} u[n]$$

Note: $nx[n] \overset{z}{\longleftrightarrow} -z\frac{d}{dz}X(z) \quad \Leftrightarrow \quad na^n \overset{z}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{z}{z-a}\right) = \frac{az}{(z-a)^2}$

Example: Suppose that $X(z) = \dfrac{6z^3 + 2z^2 - z}{z^3 - z^2 - z + 1}$

$$\frac{X(z)}{z} = \frac{6z^2 + 2z - 1}{z^3 - z^2 - z + 1} = \frac{6z^2 + 2z - 1}{(z-1)^2(z+1)}$$

$$\frac{X(z)}{z} = \frac{c_1}{z-1} + \frac{c_2}{(z-1)^2} + \frac{c_3}{z+1}$$

$$c_2 = \left[(z-1)^2 \frac{X(z)}{z}\right]_{z=1} = \frac{6+2-1}{2} = 3.5$$

$$c_1 = \left[\frac{d}{dz}(z-1)^2 \frac{X(z)}{z}\right]_{z=1} = \left[\frac{d}{dz}\frac{6z^2+2z-1}{z+1}\right]_{z=1}$$

$$= \left.\frac{(z+1)(12z+2)-(6z^2+2z-1)(1)}{(z+1)^2}\right|_{z=1}$$

$$= \frac{2(14)-7}{4} = 5.25$$

$$c_3 = \left[(z+1)\frac{X(z)}{z}\right]_{z=-1} = \frac{6-2-1}{(-2)^2} = 0.75$$

$$X(z) = \frac{5.25z}{z-1} + \frac{3.5z}{(z-1)^2} + \frac{0.75z}{z+1}$$

$$x[n] = 5.25(1)^n + 3.5n(1)^{n-1} + 0.75(-1)^n, \quad n = 0, 1, 2, \ldots$$

$$= 5.25 + 3.5n + 0.75(-1)^n, \qquad n = 0, 1, 2, \ldots$$

Using MATLAB:

```
num = [6 2 -1];
den = [1 -1 -1 1];
[r,p] = residue(num,den)
```

```
r =           p =
   5.2500        1.0000
   3.5000        1.0000
   0.7500       -1.0000
```

```
num = [6 2 -1 0];
den = [1 -1 -1 1];
x = filter(num,den,[1 zeros(1,19)]);
```

Pole Location and the Form of a Signal:

Distinct real poles:

$$x[n] = c_0\delta[n] + c_1p_1^n + c_2p_2^n + \cdots + c_Np_N^n, \quad n = 0, 1, 2, \ldots$$

Distinct complex poles:

$$x[n] = c_0\delta[n] + 2|c_1|\sigma^n \cos(\Omega n + \angle c_1) + c_3p_3^n + \cdots + c_np_N^n, n = 0, 1, 2, \ldots$$

Repeated real poles:

$$c_2n(p_1)^{n-1}u[n] \leftrightarrow \frac{c_2z}{(z - p_1)^2} \qquad \frac{1}{2}c_3n(n-1)(p_1)^{n-2}u[n] \leftrightarrow \frac{c_3z}{(z - p_1)^3}$$

Repeated complex poles:

$$c_1\sigma^n \cos(\Omega n + \theta_1) + c_2n\sigma^n \cos(\Omega n + \theta_2)$$