EE 362: Signals and Systems

The z-Transform and Discrete-Time Systems – Chapter 7

Lecture 18

Chapter Contents

- Z-Transform of a Discrete-Time Signal
- Properties of the z-transform
- Computation of the inverse z-transform
- Transfer Function Representation
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Z-Transform of a Discrete-Time Signal

- The z-transform is the discrete-time counterpart of the Laplace transform.
- Given a discrete-time signal $x[n]$, the discrete-time Fourier transform (DTFT) is defined by

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$X(\Omega)$ is in general a complex-valued function of the frequency variable Ω.

- The z-transform of the signal $x[n]$ is generated by the addition of the factor ρ^{-n}.

- The factor ρ^{-n} plays the same role as the exponential factor $e^{-\sigma t}$ that was added to the Fourier transform to generate the Laplace transform in the continuous-time case. Inserting ρ^{-n} gives,

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\rho^{-n}e^{-j\Omega n}$$

Or,

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n](\rho e^{j\Omega})^{-n}$$

This is a function in the complex number $z = \rho e^{j\Omega}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- The function $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ is the **two-sided** z-transform of the discrete-time signal $x[n]$.
- The **one-sided** z-transform of $x[n]$, also denoted $X(\Omega)$, is defined by

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

- The one-sided z-transform is a power series in z^{-1} whose coefficients are the values of the signal $x[n]$.
- If $x[n] = 0$ for all $n < 0$, the one- and two-sided z-transforms are identical.

- The one-sided z-transform can be applied to signals $x[n]$ that are nonzero for $n < 0$, but any nonzero values of $x[n]$ for $n < 0$ cannot be recovered from the one-sided z-transform.
- In this chapter we will study the one-sided z-transform only.
- Given a discrete-time signal $x[n]$ with z-transform $X(z)$, the set of all complex numbers z such that the summation converges is called the **region of convergence** of the z-transform $X(z)$.
- The z-transform exists (is well defined) for any value z belonging to the region of convergence.

Example 1: Let $\delta[n]$ denote the unit pulse concentrated at $n = 0$ given by,

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

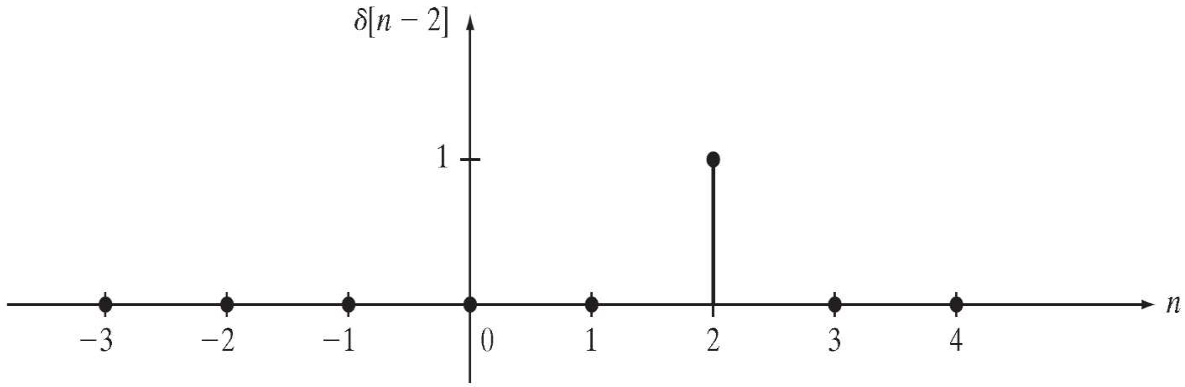
Since $\delta[n]$ is zero for all n except $n = 0$, the z-transform is

$$\sum_{n=0}^{\infty} \delta[n] z^{-n} = \delta[0] z^{-0} = 1$$

Thus, the z-transform of the unit pulse $\delta[n]$ is 1.

Since the summation exists for any value of z, the region of convergence of the z-transform of the unit pulse is the set of all complex numbers.

Example 2: Given a positive integer q, consider a unit pulse $\delta[n-q]$ located at $n=q$. For example, when $q=2$, $\delta[n-2]$ is shown below.



For any positive integer value of q, the z-transform of $\delta[n-q]$ is,

$$\sum_{n=0}^{\infty}\delta[n-q]z^{-n}=\delta[0]z^{-q}=z^{-q}=\frac{1}{z^{q}}$$

The region of convergence is all complex numbers z such that z is not zero.

Example 3: Consider the discrete-time unit-step function $u[n]$ given by

$$u[n] = \begin{cases} 1, & n = 0, 1, 2, \ldots \\ 0, & n = -1, -2, \ldots \end{cases}$$

The z-transform $U(z)$ is

$$U(z) = \sum_{n=0}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$

In the rational form

$$(z-1)U(z) = (z + 1 + z^{-1} + z^{-2} + \cdots) - (1 + z^{-1} + z^{-2} + \cdots)$$

$$= z$$

Or

$$U(z) = \frac{z}{z-1} = \frac{1}{1 - z^{-1}}$$

- Note that the form of the z-transform $U(z)$ of the discrete-time unit-step function $u[n]$ is different from the form of the Laplace transform $1/s$ of the continuous-time unit-step function $u(t)$.
- The region of convergence for the z-transform of the unit-step function includes the set of all complex numbers z such that $|z| > 1$.

$$\left| \sum_{n=0}^{\infty} (1) z^{-n} \right| < \infty \qquad \text{if } |z| > 1$$

$$\sum_{n=0}^{q} z^{-n} = \frac{(1/z)^{q+1} - 1}{(1/z) - 1} \qquad \text{where} \quad z \neq 1$$

$$\left|\sum_{n=0}^{q} z^{-n}\right| = \left|\frac{(1/z)^{q+1} - 1}{(1/z) - 1}\right| \leq \frac{(1/|z|)^{q+1} + 1}{|(1/z) - 1|}$$

If $|z| > 1$, yields,

$$\left|\sum_{n=0}^{\infty} z^{-n}\right| = \lim_{q\to\infty}\left|\sum_{n=0}^{q} z^{-n}\right| \leq \frac{1}{|(1/z) - 1|} < \infty$$

Then,

$$\left|\sum_{n=0}^{\infty} (1) z^{-n}\right| < \infty \qquad \text{if } |z| > 1$$

is verified.

Example 4: Given a real or complex number a, let $x[n] = a^n u[n]$. The z-transform $X(z)$ of $x[n]$ is given by

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= 1 + az^{-1} + a^2 z^{-2} + \cdots$$

In the rational form,

$$(z - a)X(z) = (z + a + a^2 z^{-1} + a^3 z^{-2} + \cdots) - (a + a^2 z^{-1} + a^3 z^{-2} + \cdots)$$

$$= z$$

Or, $$X(z) = \frac{z}{z - a} = \frac{1}{1 - az^{-1}}$$

Region of convergence: all complex numbers such that $|z| > |a|$

Relationship Between the DTFT & the z-Transform

The z-transform $X(z)$ of a discrete-time signal $x[n]$ can be viewed as a generalization of the discrete-time Fourier transform (DTFT) $X(\Omega)$. In other words,

$$X(\Omega) = X(z)\big|_{z=e^{j\Omega}}$$

This relationship is not valid in general unless the region of convergence of $X(z)$ includes all complex numbers z such that $|z| = 1$. For example, if $|a| < 1$,

$$X(\Omega) = X(z)\big|_{z=e^{j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega} - a} = \frac{1}{1 - ae^{-j\Omega}}$$

Transfer Notation:

Given a signal $x[n]$ with z-transform $X(z)$, the following transform pair notation is used,

$$x[n] \leftrightarrow X(z)$$

Some z-transform pairs are shown below.

Table 7.1 Basic z-Transform pairs

$\delta[n] \leftrightarrow 1$
$\delta[n-q] \leftrightarrow \frac{1}{z^q} = z^{-q}$
$u[n] \leftrightarrow \frac{z}{z-1} = \frac{1}{1-z^{-1}}$
$a^n u[n] \leftrightarrow \frac{z}{z-a} = \frac{1}{1-az^{-1}}$, a real or complex

Linearity: The z-transform is a linear operation.

If: $\begin{cases} x[n] \overset{z}{\longleftrightarrow} X(z) \\ v[n] \overset{z}{\longleftrightarrow} V(z) \end{cases}$

Then: $ax[n] + bv[n] \leftrightarrow aX(z) + bV(z)$

where *a*, *b* are constants

Example: Let $x[n] = u[n]$ and $v[n] = a^n u[n]$ where $a \neq 1$

Find the z-transform of $x[n] + v[n]$.

$$u[n] \leftrightarrow \frac{z}{z-1} \quad \text{and} \quad a^n u[n] \leftrightarrow \frac{z}{z-a}$$

$$u[n] + a^n u[n] \leftrightarrow \frac{z}{z-1} + \frac{z}{z-a} = \frac{2z^2 - (1+a)z}{(z-1)(z-a)}$$

Right shift of $x[n]u[n]$:

$$x[n-q]u[n-q] \leftrightarrow z^{-q}X(z)$$

where q is a positive integer number.

Proof: By z-transform definition,

$$x[n-q]u[n-q] \leftrightarrow \sum_{n=0}^{\infty} x[n-q]u[n-q]z^{-n}$$

Since $u[n-q]=1$ for $n \geq q$ and $u[n-q]=0$ for $n<q$,

$$x[n-q]u[n-q] \leftrightarrow \sum_{n=q}^{\infty} x[n-q]z^{-n}$$

Consider change of index. Let $\bar{n}=n-q$ so that $n=\bar{n}+q$.

$$\sum_{n=q}^{\infty} x[n-q]z^{-n} = \sum_{\bar{n}=0}^{\infty} x[\bar{n}]z^{-(\bar{n}+q)} = z^{-q}\sum_{\bar{n}=0}^{\infty} x[\bar{n}]z^{-\bar{n}} = z^{-q}X(z)$$

Example: Find the z-transform of the pulse $p[n]$ defined by

$$p[n] = \begin{cases} 1, & n = 0, 1, 2, \ldots, q-1 \\ 0, & \text{all other } n \end{cases}$$

It can be expressed as, $p[n] = u[n] - u[n-q]$

$$u[n] \overset{z}{\longleftrightarrow} \frac{z}{z-1}$$

$$u[n-p] \overset{z}{\longleftrightarrow} \frac{z}{z-1}z^{-p} = \frac{z^{-p+1}}{z-1}$$

$$\frac{z}{z-1} - \frac{z^{-q+1}}{z-1} = \frac{z(1-z^{-q})}{z-1} = \frac{z^q - 1}{z^{q-1}(z-1)}$$

Right shift of $x[n]$: From z-transform,

$$x[n-1] \leftrightarrow \sum_{n=0}^{\infty} x[n-1]z^{-n}$$ Change index: $\bar{n} = n - 1$

$$x[n-1] \leftrightarrow \sum_{\bar{n}=-1}^{\infty} x[\bar{n}]z^{-(\bar{n}+1)} = \sum_{\bar{n}=0}^{\infty} x[\bar{n}]z^{-(\bar{n}+1)} + x[-1]$$

$$\leftrightarrow z^{-1} \sum_{\bar{n}=0}^{\infty} x[\bar{n}]z^{-\bar{n}} + x[-1]$$

$$\leftrightarrow z^{-1}X(z) + x[-1]$$

If $x[n] = 0$ for $n = -1, -2, \ldots, -q$, the transform becomes

$$x[n-q] \leftrightarrow z^{-q}X(z)$$

which is identical to $x[n-q]u[n-q] \leftrightarrow z^{-q}X(z)$

Now, suppose that $x[n] \leftrightarrow X(z)$. Then,

$$x[n-1] \leftrightarrow z^{-1}X(z) + x[-1]$$

$$x[n-2] \leftrightarrow z^{-2}X(z) + x[-2] + z^{-1}x[-1]$$

$$\vdots$$

$$x[n-q] \leftrightarrow z^{-q}X(z) + x[-q] + z^{-1}x[-q+1] + \cdots + z^{-q+1}x[-1]$$

Example:

$$x[n-1] \overset{z}{\longleftrightarrow} z^{-1}X(z) + x[-1]$$

$$x[n-2] \overset{z}{\longleftrightarrow} z^{-2}X(z) + x[-1]z^{-1} + x[-2]$$

Left shift in time: Given a discrete-time signal $x[n]$ and a positive integer q, the q-step left shift of $x[n]$ is the signal $x[n+q]$.

$$x[n+1] \leftrightarrow zX(z) - x[0]z$$

$$x[n+2] \leftrightarrow z^2X(z) - x[0]z^2 - x[1]z$$

$$\vdots$$

$$x[n+q] \leftrightarrow z^qX(z) - x[0]z^q - x[1]z^{q-1} - \cdots - x[q-1]z$$

Proof: from z-transform,

$$x[n+1] \leftrightarrow \sum_{n=0}^{\infty} x[n+1]z^{-n}$$

Change index: $\bar{n} = n + 1$

$$x[n+1] \leftrightarrow \sum_{\bar{n}=1}^{\infty} x[\bar{n}] z^{-(\bar{n}-1)}$$

$$\leftrightarrow z \sum_{\bar{n}=1}^{\infty} x[\bar{n}] z^{-\bar{n}} = z\left[\sum_{\bar{n}=0}^{\infty} x[\bar{n}] z^{-\bar{n}} - x[0]\right]$$

$$\leftrightarrow z[X(z) - x[0]]$$

Example: find the z-transform of $u[n+1]$.

Using the left shift property, z-transform of $u[n+1]$ is given by

$$zU(z) - u[0]z = \frac{z^2}{z-1} - z = \frac{z^2 - z(z-1)}{z-1} = \frac{z}{z-1}$$

Then z-transforms of $u[n]$ and $u[n+1]$ are the same because $u[n+1] = u[n]$ for $n \geq 0$.

Multiplication by n and n^2: if $x[n] \leftrightarrow X(z)$, then

$$nx[n] \leftrightarrow -z\frac{d}{dz}X(z)$$

$$n^2x[n] \leftrightarrow z\frac{d}{dz}X(z) + z^2\frac{d^2}{dz^2}X(z)$$

Proof:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

$$\frac{d}{dz}X(z) = \sum_{n=0}^{\infty}(-n)x[n]z^{-n-1}$$

$$= -z^{-1}\sum_{n=0}^{\infty} nx[n]z^{-n}$$

$$\Rightarrow \quad -z\frac{d}{dz}X(z) = \sum_{n=0}^{\infty} nx[n]z^{-n}$$

Example: Compute the z-transform of the signal $na^n u[n]$. Let $x[n] = a^n u[n]$.

From table 7.1, $X(z) = \dfrac{z}{z-a}$, then

$$z\frac{d}{dz}X(z) = z\left[\frac{-z}{(z-a)^2} + \frac{1}{z-a}\right] = \frac{-az}{(z-a)^2}$$

which gives the transform pair,

$$na^n u[n] \leftrightarrow \frac{az}{(z-a)^2}$$

If $a = 1$,

$$nu[n] \leftrightarrow \frac{z}{(z-1)^2}$$

Example: Compute the z-transform of the signal $n^2 a^n u[n]$. Let $x[n] = a^n u[n]$ so that $X(z) = z/(z-a)$. Then,

$$\frac{d^2}{dz^2} X(z) = \frac{2a}{(z-a)^3}$$

Using $n^2 x[n] \leftrightarrow z\frac{d}{dz}X(z) + z^2\frac{d^2}{dz^2}X(z)$, and $z\frac{d}{dz}X(z) = \frac{-az}{(z-a)^2}$

$$n^2 a^n u[n] \leftrightarrow \frac{-az}{(z-a)^2} + \frac{2az^2}{(z-a)^3}$$

$$\leftrightarrow \frac{az(z+a)}{(z-a)^3}$$

If $a = 1$, $n^2 u[n] \leftrightarrow \frac{z(z+1)}{(z-1)^3}$

Multiplication by a^n: if $x[n] \leftrightarrow X(z)$, then for any nonzero real or complex number a,

$$a^n x[n] \leftrightarrow X\left(\frac{z}{a}\right)$$

Proof: observe that

$$a^n x[n] \leftrightarrow \sum_{n=0}^{\infty} a^n x[n] z^{-n}$$

$$\leftrightarrow \sum_{n=0}^{\infty} x[n] \left(\frac{z}{a}\right)^{-n} = X\left(\frac{z}{a}\right)$$

Example: Let $p[n] = u[n] - u[n-q]$ where q is a positive integer, compute z-transform of $a^n p[n]$.

The z-transform of $p[n]$ is as follows,

$$\frac{z(1 - z^{-q})}{z - 1}$$

Then, from $a^n x[n] \leftrightarrow X\left(\frac{z}{a}\right)$,

$$a^n p[n] \leftrightarrow \frac{(z/a)[1 - (z/a)^{-q}]}{(z/a) - 1}$$

$$\leftrightarrow \frac{z(1 - a^q z^{-q})}{z - a}$$

Multiplication by $\cos(\Omega n)$ and $\sin(\Omega n)$:

$$(\cos \Omega n)x[n] \leftrightarrow \frac{1}{2}[X(e^{j\Omega}z) + X(e^{-j\Omega}z)]$$

$$(\sin \Omega n)x[n] \leftrightarrow \frac{j}{2}[X(e^{j\Omega}z) - X(e^{-j\Omega}z)]$$

Proof: using Euler's formula

$$(\cos \Omega n)x[n] = \frac{1}{2}[e^{-j\Omega n}x[n] + e^{j\Omega n}x[n]]$$

$$(\sin \Omega n)x[n] = \frac{j}{2}[e^{-j\Omega n}x[n] - e^{j\Omega n}x[n]]$$

Using $a^n x[n] \leftrightarrow X\left(\frac{z}{a}\right)$ we get

$$e^{-j\Omega n}x[n] \leftrightarrow X(e^{j\Omega}z) \quad \text{and} \quad e^{j\Omega n}x[n] \leftrightarrow X(e^{-j\Omega}z)$$

Example: compute z-transform of $v[n] = u[n]\cos(\Omega n)$. Let $x[n] = u[n]$ so that $X(z) = z/(z-1)$. Then,

$$(\cos \Omega n)u[n] \leftrightarrow \frac{1}{2}\left(\frac{e^{j\Omega}z}{e^{j\Omega}z - 1} + \frac{e^{-j\Omega}z}{e^{-j\Omega}z - 1}\right)$$

$$\leftrightarrow \frac{1}{2}\left[\frac{e^{j\Omega}z(e^{-j\Omega}z - 1) + e^{-j\Omega}z(e^{j\Omega}z - 1)}{(e^{j\Omega}z - 1)(e^{-j\Omega}z - 1)}\right]$$

$$\leftrightarrow \frac{1}{2}\left[\frac{z^2 - e^{j\Omega}z + z^2 - e^{-j\Omega}z}{z^2 - (e^{j\Omega} + e^{-j\Omega})z + 1}\right]$$

$$\leftrightarrow \frac{z^2 - (\cos \Omega)z}{z^2 - (2\cos \Omega)z + 1}$$

Similarly,

$$(\sin \Omega n)u[n] \leftrightarrow \frac{(\sin \Omega)z}{z^2 - (2\cos \Omega)z + 1}$$

Example: compute z-transform of $v[n] = a^n u[n]\cos(\Omega n)$. Let $x[n] = u[n]\cos(\Omega n)$ and apply the multiplication by a^n,

$$a^n(\cos\ \Omega n)u[n] \leftrightarrow \frac{(z/a)^2 - (\cos\ \Omega)(z/a)}{(z/a)^2 - (2\cos\ \Omega)(z/a) + 1}$$

$$\leftrightarrow \frac{z^2 - (a\cos\ \Omega)z}{z^2 - (2a\cos\ \Omega)z + a^2}$$

Similarly

$$a^n(\sin\ \Omega n)u[n] \leftrightarrow \frac{(a\sin\ \Omega)z}{z^2 - (2a\cos\ \Omega)z + a^2}$$

Summation: let $x[n]$ be zero for $n < 0$ and $v[n]$ denote the sum of $x[n]$ defined by:

$$v[n] = \sum_{i=0}^{n} x[i] \quad \leftrightarrow \quad \frac{z}{z-1} X(z)$$

Proof:

$$v[n] = \sum_{i=0}^{n} x[i] = \sum_{i=0}^{n-1} x[i] + x[n] = v[n-1] + x[n]$$

$$V(z) = z^{-1} V(z) + X(z)$$

$$V(z) = \frac{1}{1 - z^{-1}} X(z) = \frac{z}{z-1} X(z)$$

Example: Compute z-transform of $(n+1)u[n]$.

Let $x[n] = u[n]$. Then the sum

$$v[n] = \sum_{i=0}^{n} u[i] = (n+1)u[n]$$

Then

$$V(z) = \frac{z}{z-1} X(z) = \frac{z^2}{(z-1)^2}$$

Thus,

$$(n+1)u[n] \leftrightarrow \frac{z^2}{(z-1)^2}$$

Convolution: like the Fourier transform, the z-transform of the convolution $x*v$ is equal to the product $X(z)V(z)$.

Proof: $$x[n]*v[n] = \sum_{i=0}^{n} x[i]v[n-i]$$

$$x[n]*v[n] \leftrightarrow \sum_{n=0}^{\infty}\left[\sum_{i=0}^{\infty} x[i]v[n-i]\right]z^{-n}$$

$$\leftrightarrow \sum_{i=0}^{\infty} x[i]\left[\sum_{n=0}^{\infty} v[n-i]z^{-n}\right]$$

Change index:

$$\bar{n} = n - i$$

$$x[n]*v[n] \leftrightarrow \sum_{i=0}^{\infty} x[i]\left[\sum_{\bar{n}=-i}^{\infty} v[\bar{n}]z^{-\bar{n}-i}\right]$$

$$\leftrightarrow \sum_{i=0}^{\infty} x[i]\left[\sum_{\bar{n}=0}^{\infty} v[\bar{n}]z^{-\bar{n}-i}\right], \text{ since } v[\bar{n}] = 0 \text{ for } \bar{n} < 0$$

$$\leftrightarrow \left[\sum_{i=0}^{\infty} x[i]z^{-i}\right]\left[\sum_{\bar{n}=0}^{\infty} v[\bar{n}]z^{-\bar{n}}\right]$$

$$\leftrightarrow X(z)V(z)$$

Initial-Value Theorem:

$$x[0] = \lim_{z \to \infty} X(z)$$

$$x[1] = \lim_{z \to \infty} [zX(z) - zx[0]]$$

$$\vdots$$

$$x[q] = \lim_{z \to \infty} [z^q X(z) - z^q x[0] - z^{q-1} x[1] - \cdots - zx[q-1]]$$

Proof:

$$z^{-n} \to 0 \qquad \text{as } z \to \infty \qquad \text{for all } n \geq 1$$

$$x[n]z^{-n} \to 0 \qquad \text{as } z \to \infty \qquad \text{for all } n \geq 1$$

Taking the limit as $z \to \infty$ of both sides of $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$ produces the results shown above.

Final-Value Theorem: Given a discrete-time signal $x[n]$ with z-transform $X(z)$, suppose that $x[n]$ has a limit as $n \to \infty$. Then, the final value theorem states that,

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$$

The final value theorem is not always applicable.

Example: Suppose that,

$$X(z) = \frac{3z^2 - 2z + 4}{z^3 - 2z^2 + 1.5z - 0.5}$$

In this case $X(z)$ has a pole at $z = 1$, and thus there is a pole-zero cancellation in $(z-1)X(z)$. Performing the cancellation gives,

$$X(z) = \frac{3z^2 - 2z + 4}{z^3 - 2z^2 + 1.5z - 0.5} \qquad (z-1)X(z) = \frac{3z^2 - 2z + 4}{z^2 - z + 0.5}$$

Using MATLAB command "roots" reveals that the poles of $(z-1)X(z)$ are $z = 0.5 \pm j0.5$. The magnitude of both these poles is equal to 0.707, and therefore $x[n]$ has a limit as $n \to \infty$.

$$\lim_{n\to\infty} x[n] = [(z-1)X(z)]_{z=1} = \left[\frac{3z^2 - 2z + 4}{z^2 - z + 0.5}\right]_{z=1} = \frac{5}{0.5} = 10$$

$$ax[n] + bv[n] \leftrightarrow aX(z) + bV(z)$$

$$x[n-q]u[n-q] \leftrightarrow z^{-q}X(z)$$

$$x[n-1] \leftrightarrow z^{-1}X(z) + x[-1]$$

$$x[n-2] \leftrightarrow z^{-2}X(z) + x[-2] + z^{-1}x[-1]$$

$$\vdots$$

$$x[n-q] \leftrightarrow z^{-q}X(z) + x[-q] + z^{-1}x[-q+1] + \cdots + z^{-q+1}x[-1]$$

$$x[n+1] \leftrightarrow zX(z) - x[0]z$$

$$x[n+2] \leftrightarrow z^2X(z) - x[0]z^2 - x[1]z$$

$$x[n+q] \leftrightarrow z^qX(z) - x[0]z^q - x[1]z^{q-1} - \cdots - x[q-1]z$$

$$nx[n] \leftrightarrow -z\frac{d}{dz}X(z)$$

$$n^2x[n] \leftrightarrow z\frac{d}{dz}X(z) + z^2\frac{d^2}{dz^2}X(z)$$

$$a^nx[n] \leftrightarrow X\left(\frac{z}{a}\right)$$

$$(\cos \Omega n)x[n] \leftrightarrow \frac{1}{2}[X(e^{j\Omega}z) + X(e^{-j\Omega}z)]$$

$$(\sin \Omega n)x[n] \leftrightarrow \frac{j}{2}[X(e^{j\Omega}z) - X(e^{-j\Omega}z)]$$

$$\sum_{i=0}^{n} x[i] \leftrightarrow \frac{z}{z-1}X(z)$$

$$x[n]*v[n] \leftrightarrow X(z)V(z)$$

$$x[0] = \lim_{z\to\infty} X(z)$$

$$x[1] = \lim_{z\to\infty} [zX(z) - zX[0]]$$

$$\vdots$$

$$x[q] = \lim_{z\to\infty} [z^qX(z) - z^qx[0] - z^{q-1}x[1] - \cdots - zx[q-1]]$$

If X(z) is rational and the poles of (z-1)X(z) have magnitudes < 1, then

$$\lim_{n\to\infty} x[n] = [(z-1)X(z)]_{z=1}$$

$$\delta[n] \leftrightarrow 1$$

$$\delta[n-q] \leftrightarrow \frac{1}{z^q}, \quad q = 1, 2, \ldots$$

$$u[n] \leftrightarrow \frac{z}{z-1}$$

$$u[n] - u[n-q] \leftrightarrow \frac{z^q - 1}{z^{q-1}(z-1)}, \quad q = 1, 2, \ldots$$

$$a^nu[n] \leftrightarrow \frac{z}{z-a}, \; a \text{ real or complex}$$

$$nu[n] \leftrightarrow \frac{z}{(z-1)^2}$$

$$(n+1)u[n] \leftrightarrow \frac{z^2}{(z-1)^2}$$

$$n^2u[n] \leftrightarrow \frac{z(z+1)}{(z-1)^3}$$

$$na^nu[n] \leftrightarrow \frac{az}{(z-a)^2}$$

$$n^2a^nu[n] \leftrightarrow \frac{az(z+a)}{(z-a)^3}$$

$$n(n+1)a^nu[n] \leftrightarrow \frac{2az^2}{(z-a)^3}$$

$$(\cos \Omega n)u[n] \leftrightarrow \frac{z^2 - (\cos \Omega)z}{z^2 - (2\cos \Omega)z + 1}$$

$$(\sin \Omega n)u[n] \leftrightarrow \frac{(\sin \Omega)z}{z^2 - (2\cos \Omega)z + 1}$$

$$a^n(\cos \Omega n)u[n] \leftrightarrow \frac{z^2 - (a\cos \Omega)z}{z^2 - (2a\cos \Omega)z + a^2}$$

$$a^n(\sin \Omega n)u[n] \leftrightarrow \frac{(a\sin \Omega)z}{z^2 - (2a\cos \Omega)z + a^2}$$