EE 362: Signals and Systems

Laplace Transform & Transfer Function Representation – Chapter 6

Lecture 17

Chapter Contents

- Laplace transform of a signal
- Properties of Laplace transform
- Computation of inverse Laplace transform
- Transform of input/output differential equation
- Transform of input/output convolution integral
- Direct construction of the transfer function

Transform of input/output differential equation

- We can generate an "s-domain description" of systems defined by input/output differential equations by taking the Laplace transform.
- This yields an algebraic procedure to solve the input/output differential equation.
- **First-Order Case**:

Consider the linear time-invariant continuous-time system given by the first-order input/output differential equation

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

where a and b are real numbers, $y(t)$ is the output, and $x(t)$ is the input.

- Taking the Laplace transform gives,

$$sY(s) - y(0^-) + aY(s) = bX(s)$$

- Rearranging terms gives,

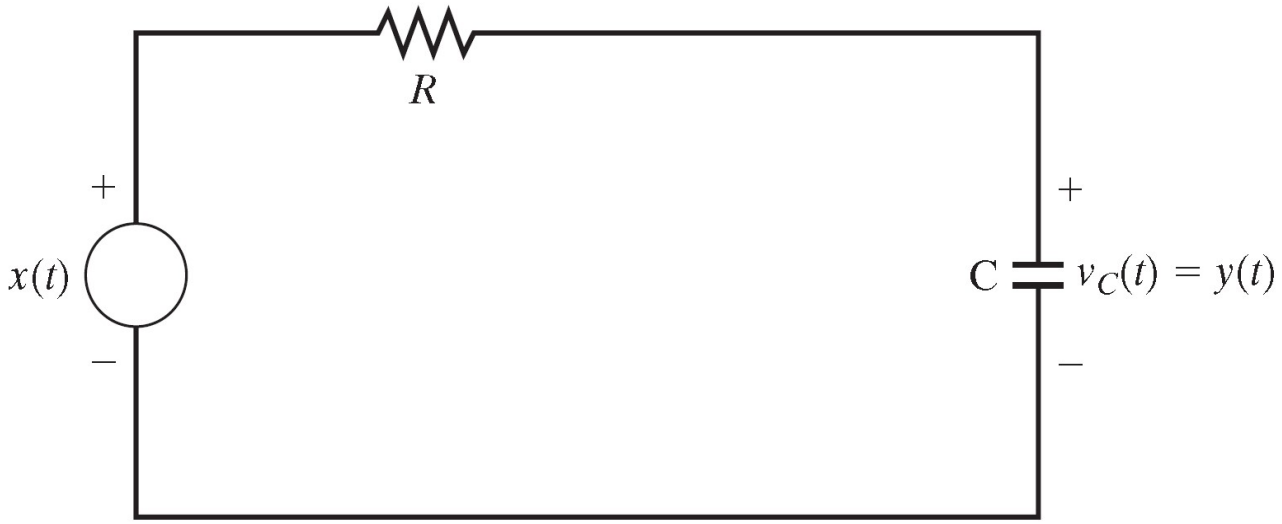
$$(s + a)Y(s) = y(0^-) + bX(s)$$

$$Y(s) = \frac{y(0^-)}{s + a} + \frac{b}{s + a}X(s)$$

- If the initial conditions are zero,

$$Y(s) = \frac{b}{s + a}X(s) \quad \Rightarrow \quad H(s) = \frac{b}{s + a}$$

- The function $H(s)$ is called the ***transfer function*** of the system.
- ***Example***: Consider the RC circuit shown below.



- The input/output differential equation is,

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

- Taking the Laplace transform gives,

$$Y(s) = \frac{y(0^-)}{s + 1/RC} + \frac{1/RC}{s + 1/RC} X(s)$$

- If the input is a unit-step ($X(s) = 1/s$), then

$$Y(s) = \frac{y(0^-)}{s + 1/RC} + \frac{1/RC}{(s + 1/RC)s}$$

- Expanding the second term gives,

$$Y(s) = \frac{y(0^-)}{s + 1/RC} + \frac{1}{s} - \frac{1}{s + 1/RC}$$

- Then the time-domain representation is

$$y(t) = y(0^-)e^{-(1/RC)t} + 1 - e^{-(1/RC)t}, t \geq 0$$

- If the initial condition is zero, then

$$y(t) = 1 - e^{-(1/RC)t}, t \geq 0$$

- Second-Order Case: Consider the linear time-invariant continuous-time system given by the second-order input/output differential equation

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = b_1\frac{dx(t)}{dt} + b_0x(t)$$

where a_1, a_0, b_1, and b_0 are real numbers. Assuming that $x(0^-) = 0$, the Laplace transform becomes

$$s^2Y(s) - y(0^-)s - \dot{y}(0^-) + a_1[sY(s) - y(0^-)] + a_0Y(s)$$
$$= b_1sX(s) + b_0X(s)$$

- Then,

$$Y(s) = \frac{y(0^-)s + \dot{y}(0^-) + a_1 y(0^-)}{s^2 + a_1 s + a_0} + \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} X(s)$$

The first term is the Laplace transform of the part of the output response resulting from initial conditions and the second term is the Laplace transform of the part of the output response resulting from the application of the input $x(t)$ for $t \geq 0$.

- If $y(0^-) = 0$ and $\dot{y}(0^-) = 0$, then

$$Y(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} X(s) \quad \Rightarrow \quad H(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

Example: Consider the following system

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

Then,

$$H(s) = \frac{2}{s^2 + 6s + 8}$$

If the input is a unit-step ($X(s) = 1/s$) and initial conditions are zero, then,

$$Y(s) = H(s)X(s) = \frac{2}{s^2 + 6s + 8}\frac{1}{s}$$

Expanding $Y(s)$:

$$Y(s) = \frac{0.25}{s} - \frac{0.5}{s+2} + \frac{0.25}{s+4}$$

Then,

$$y(t) = 0.25 - 0.5e^{-2t} + 0.25e^{-4t}, t \geq 0$$

If the input is a unit-step ($X(s) = 1/s$) and initial conditions are: $y(0^-) = 1$ and $\dot{y}(0^-) = 2$, then using

$$Y(s) = \frac{y(0^-)s + \dot{y}(0^-) + a_1 y(0^-)}{s^2 + a_1 s + a_0} + \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} X(s)$$

gives,

$$Y(s) = \frac{s + 8}{s^2 + 6s + 8} + \frac{2}{s^2 + 6s + 8}\frac{1}{s} = \frac{s^2 + 8s + 2}{s(s^2 + 6s + 8)}$$

Expanding yields,

$$Y(s) = \frac{0.25}{s} + \frac{2.5}{s + 2} - \frac{1.75}{s + 4}$$

Thus

$$y(t) = 0.25 + 2.5e^{-2t} - 1.75e^{-4t}, t \geq 0$$

- **Nth-Order Case**: Consider the following case

$$\frac{d^N y(t)}{dt^N} + \sum_{i=0}^{N-1} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{M} b_i \frac{d^i x(t)}{dt^i}$$

- If all initial conditions of the input are zero,

$$Y(s) = \frac{C(s)}{A(s)} + \frac{B(s)}{A(s)} X(s)$$

$$B(s) = b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0$$

$$A(s) = s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0$$

- If the initial conditions of the output are zero, then

$$Y(s) = \frac{B(s)}{A(s)}X(s) = \frac{b_M s^M + \cdots + b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0}X(s)$$

$$H(s) = \frac{b_M s^M + \cdots + b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0}$$

- If the initial conditions are not zero, we cannot apply

$$Y(s) = H(s)X(s)$$

Transform of input/output convolution integral

$$y(t) = h(t) * x(t) = \int_0^t h(\lambda)x(t-\lambda)\,d\lambda, t \geq 0$$

$$Y(s) = H(s)X(s)$$

$$h(t) \leftrightarrow H(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

- Example: Suppose that the input is $x(t) = e^{-t}u(t)$ is applied to a causal linear time-invariant continuous-time system, and the output is $y(t) = 2 - 3e^{-t} + e^{-2t}\cos 2t, t \geq 0$

Then $$Y(s) = \frac{2}{s} - \frac{3}{s+1} + \frac{s+2}{(s+2)^2+4}$$

and $$X(s) = \frac{1}{s+1}$$

Using $Y(s) = H(s)X(s)$ yields,

$$H(s) = \frac{\frac{2}{s} - \frac{3}{s+1} + \frac{s+2}{(s+2)^2+4}}{\frac{1}{s+1}} = \frac{2(s+1)}{s} - 3 + \frac{(s+1)(s+2)}{(s+2)^2+4}$$

$$= \frac{[2(s+1) - 3s][(s+2)^2+4] + s(s+1)(s+2)}{s[(s+2)^2+4]} = \frac{s^2+2s+16}{s^3+4s^2+8s}$$

- **<u>Zeros and Poles</u>**: Given a finite-dimensional system with the transfer function:

$$H(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

- Or

$$H(s) = \frac{b_M (s - z_1)(s - z_2) \cdots (s - z_M)}{(s - p_1)(s - p_2) \cdots (s - p_N)}$$

where $z_1, z_2, \ldots, z_M$ are the zeros of $H(s)$ and $p_1, p_2, \ldots, p_N$ are the poles of $H(s)$.

- The location of the poles and zeros is of fundamental importance in determining the behavior of the system.

Direct construction of the transfer function

- The transfer function of a system is often determined directly from the wiring diagram of the system, so it is not always necessary to determine the impulse response or the input/output differential equation in order to generate the transfer function representation.
- **Consider the RLC circuits**

$$v(t) = Ri(t)$$

$$\frac{dv(t)}{dt} = \frac{1}{C}i(t)$$

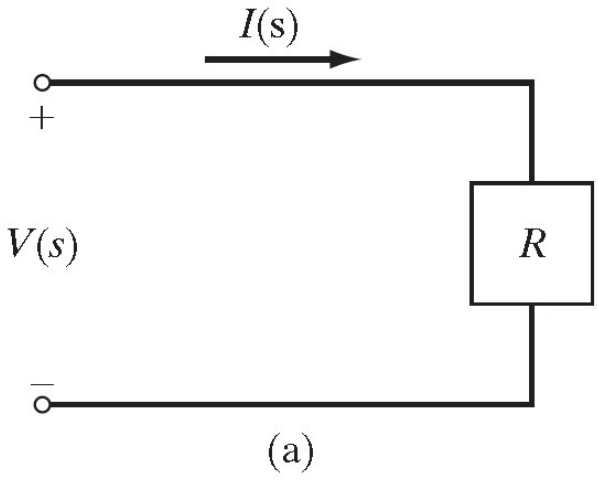
$$v(t) = L\frac{di(t)}{dt}$$

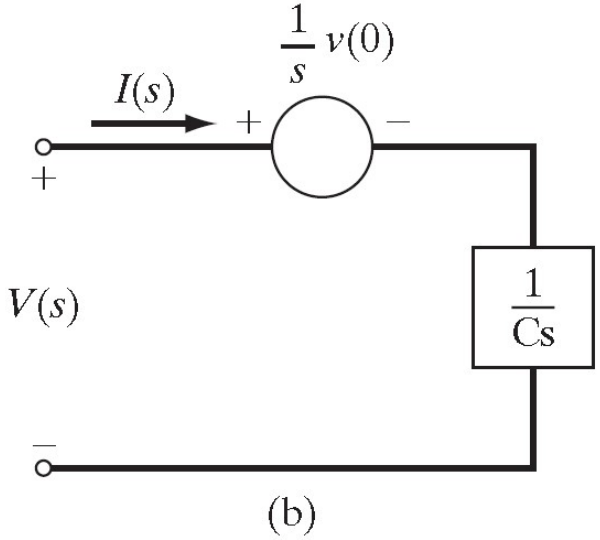
- The RLC circuit relationships can be expressed in the s-domain by taking the Laplace transform as follows,

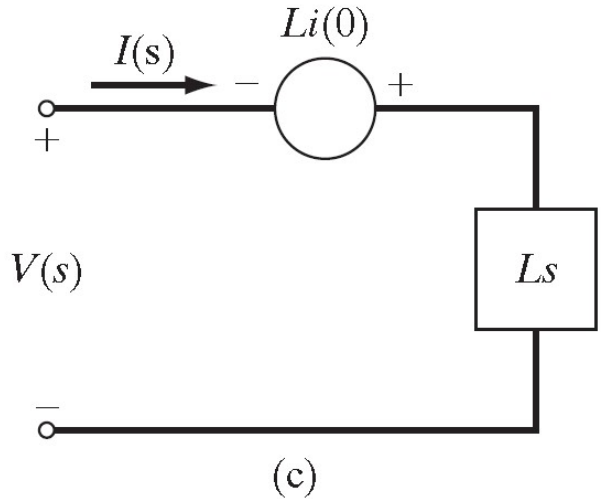
$$V(s) = RI(s)$$

$$sV(s) - v(0) = \frac{1}{C}I(s) \quad \text{or} \quad V(s) = \frac{1}{Cs}I(s) + \frac{1}{s}v(0)$$

$$V(s) = LsI(s) - Li(0)$$





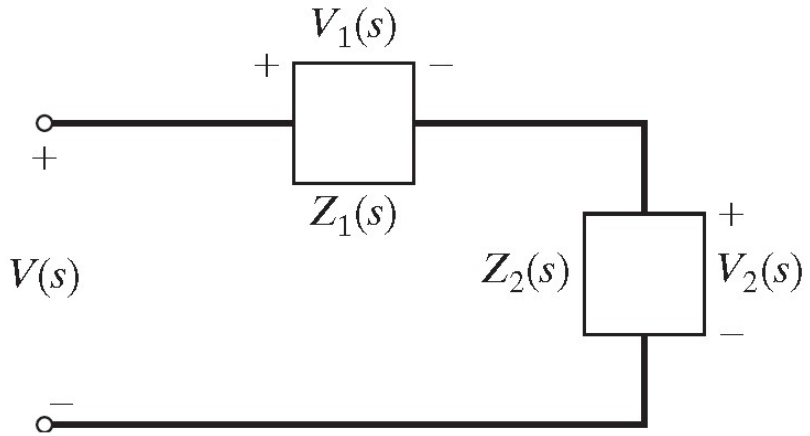


- Given an interconnection of RLCs, we can construct an s-domain representation of the circuit by taking the Laplace transform of the voltages and currents in the circuit and by expressing resistors, capacitors, and inductors in terms of their s-domain representation.
- The resulting s-domain representation satisfies the same circuit laws as purely resistive circuit with voltage and current sources.
- KVL, KCL, voltage and current divisions, etc., all can be applied.

- Consider the following circuit.

$$V_1(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} V(s)$$

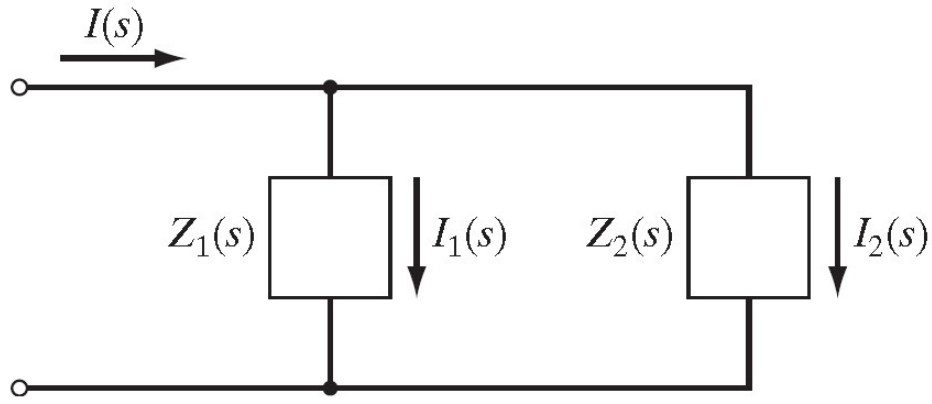
$$V_2(s) = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} V(s)$$



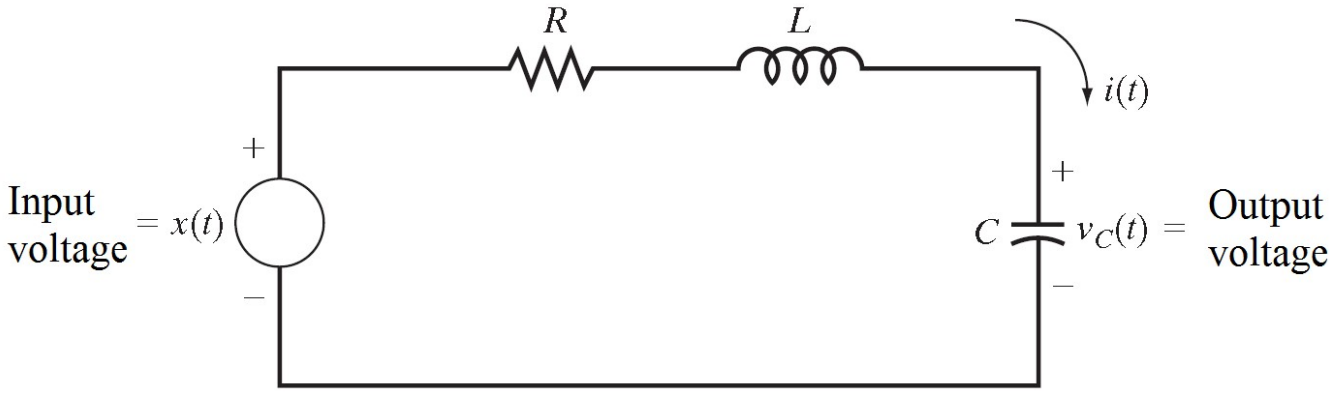
- Consider the following circuit.

$$I_1(s) = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} I(s)$$

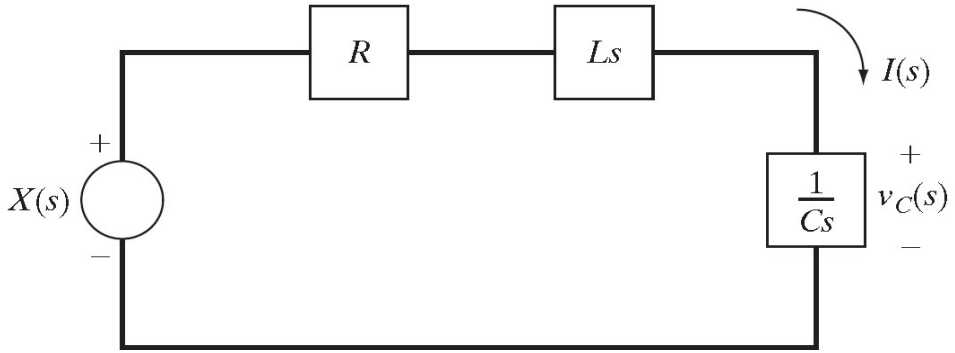
$$I_2(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} I(s)$$



Example: Consider the following RLC circuit where all initial conditions are zero.



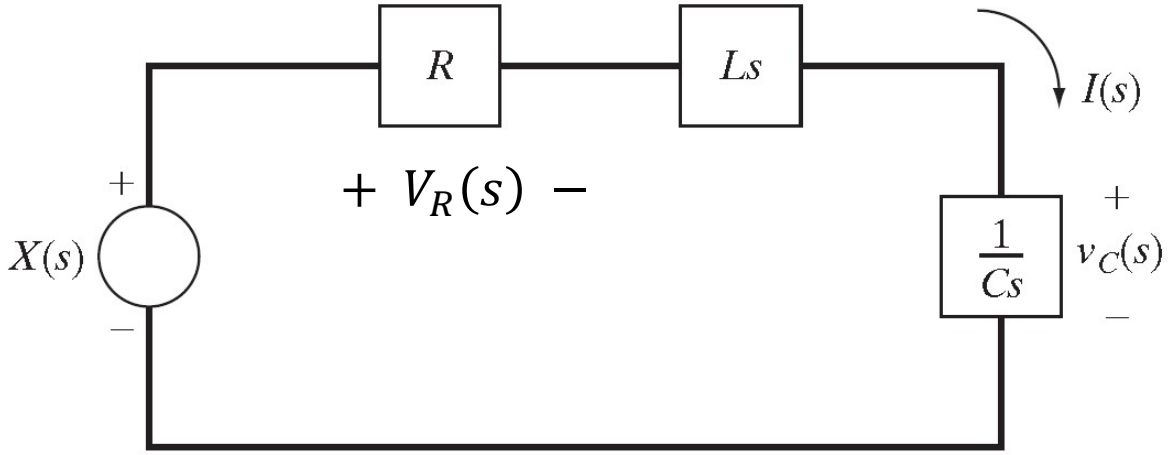
Find the transfer function.



$$V_c(s) = \frac{1/Cs}{Ls + R + (1/Cs)} X(s) = \frac{1/LC}{s^2 + (R/L)s + (1/LC)} X(s)$$

$$\Rightarrow \quad H(s) = \frac{1/LC}{s^2 + (R/L)s + (1/LC)}$$

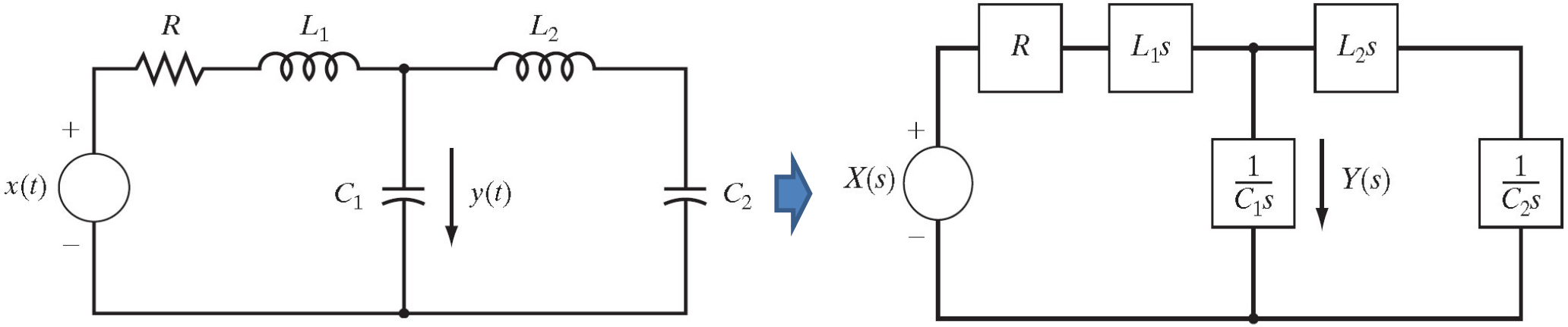
In the previous example, if the output is the voltage across the resistor, the transfer function can be found as follows,



$$V_R(s) = \frac{R}{Ls + R + (1/Cs)} X(s) = \frac{(R/L)s}{s^2 + (R/L)s + (1/LC)} X(s)$$

$$H(s) = \frac{(R/L)s}{s^2 + (R/L)s + (1/LC)}$$

Example: Consider the following RLC circuit where all initial conditions are zero.



The equivalent impedance of the parallel connection that contains C1, L2, and C2 is as follows,

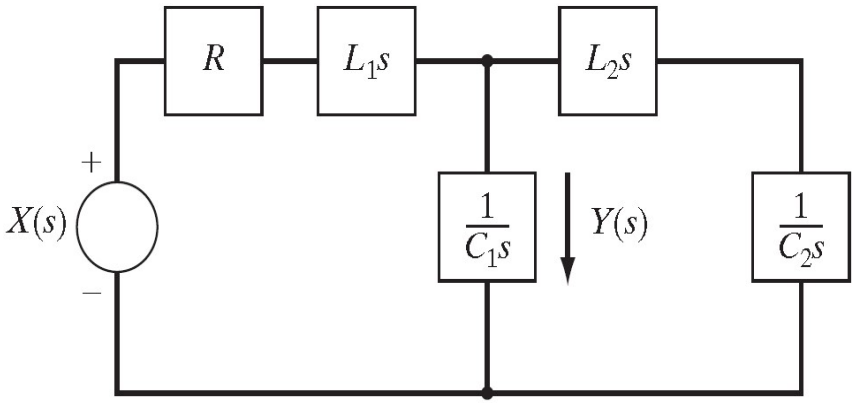
$$\frac{(1/C_1 s)[L_2 s + (1/C_2 s)]}{(1/C_1 s) + L_2 s + (1/C_2 s)} = \frac{C_2 L_2 s^2 + 1}{C_1 C_2 L_2 s^3 + (C_1 + C_2)s}$$

Let $V_1(s)$ denote the transform of the voltage across the capacitance C1. Then,

$$Y(s) = C_1 s V_1(s)$$

where

$$V_1(s) = \frac{\dfrac{C_2L_2s^2 + 1}{C_1C_2L_2s^3 + (C_1 + C_2)s}}{R + L_1s + \dfrac{C_2L_2s^2 + 1}{C_1C_2L_2s^3 + (C_1 + C_2)s}} X(s)$$



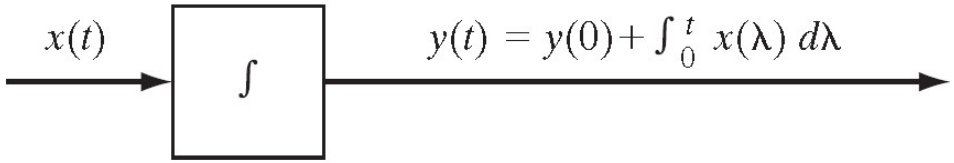
$$V_1(s) = \frac{C_2L_2s^2 + 1}{(R + L_1s)(C_1C_2L_2s^3 + (C_1 + C_2)s) + C_2L_2s^2 + 1} X(s)$$

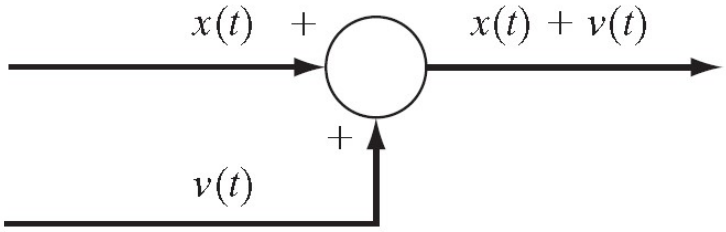
$$V_1(s) = \frac{C_2L_2s^2 + 1}{C_1C_2L_1L_2s^4 + RC_1C_2L_2s^3 + [L_1(C_1 + C_2) + L_2C_2]s^2 + R(C_1 + C_2)s + 1} X(s)$$

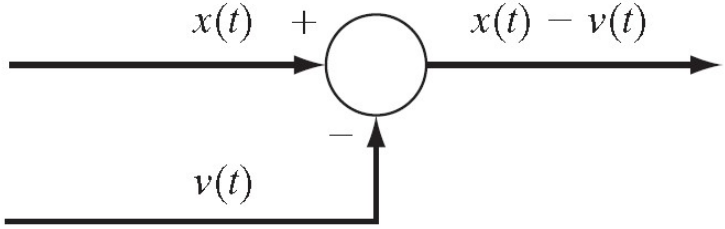
$$Y(s) = \frac{C_1C_2L_2s^3 + C_1s}{C_1C_2L_1L_2s^4 + RC_1C_2L_2s^3 + [L_1(C_1 + C_2) + L_2C_2]s^2 + R(C_1 + C_2)s + 1} X(s)$$

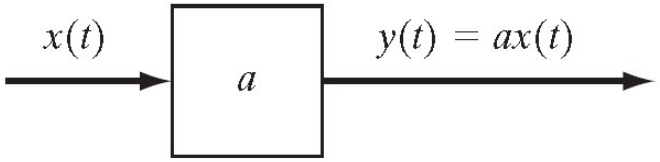
$$H(s) = \frac{C_1C_2L_2s^3 + C_1s}{C_1C_2L_1L_2s^4 + RC_1C_2L_2s^3 + [L_1(C_1 + C_2) + L_2C_2]s^2 + R(C_1 + C_2)s + 1}$$

Interconnections of Integrators: Continuous-time systems are sometimes given in terms of an interconnection of integrators, adders, subtractors, and scalar multipliers as shown below.









$$y(t) = y(0) + \int_0^t x(\lambda)\, d\lambda \quad \Rightarrow \quad \frac{dy(t)}{dt} = x(t)$$

$$\frac{dy(t)}{dt} = x(t) \quad \Rightarrow \quad sY(s) - y(0) = X(s)$$

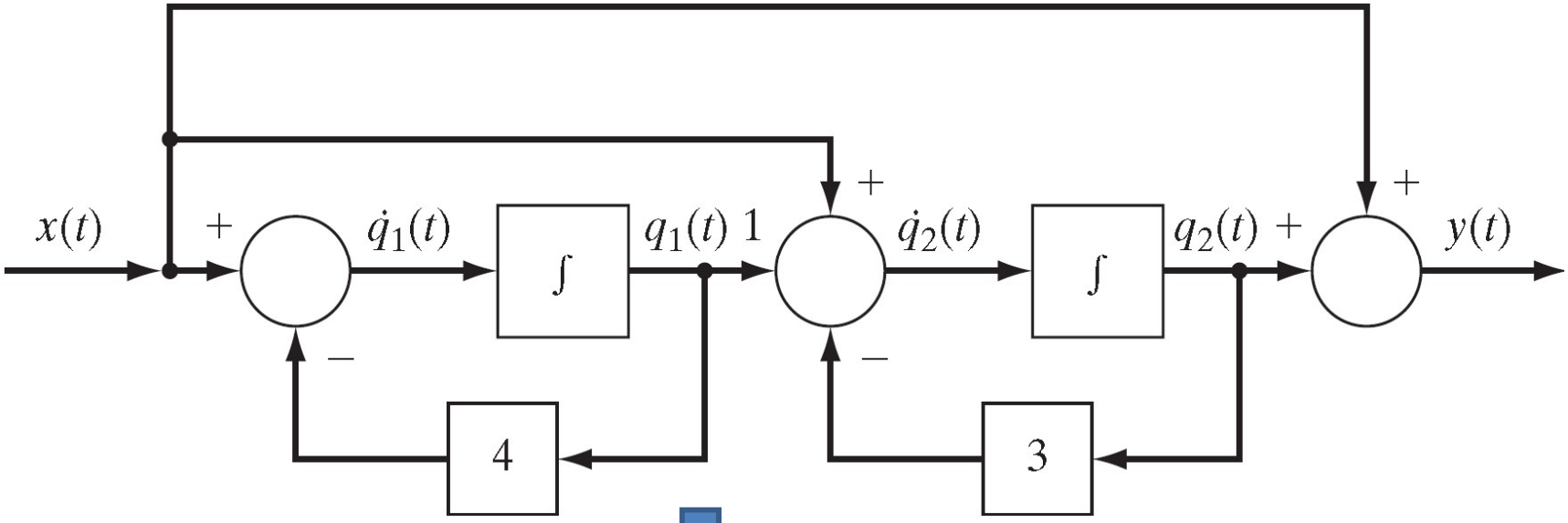
$$Y(s) = \frac{1}{s}y(0) + \frac{1}{s}X(s)$$

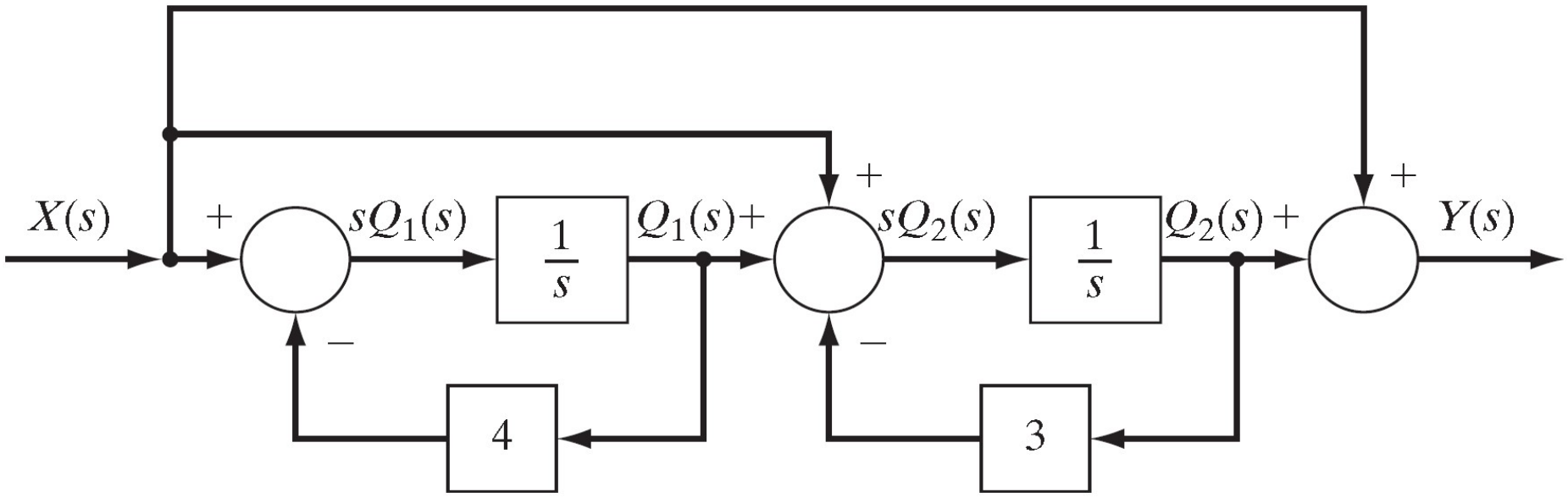
$$Y(s) = \frac{1}{s}X(s)$$

Then the integrator has the transfer function

$$H_{int}(s) = \frac{1}{s}$$

Example: Determine the transfer function

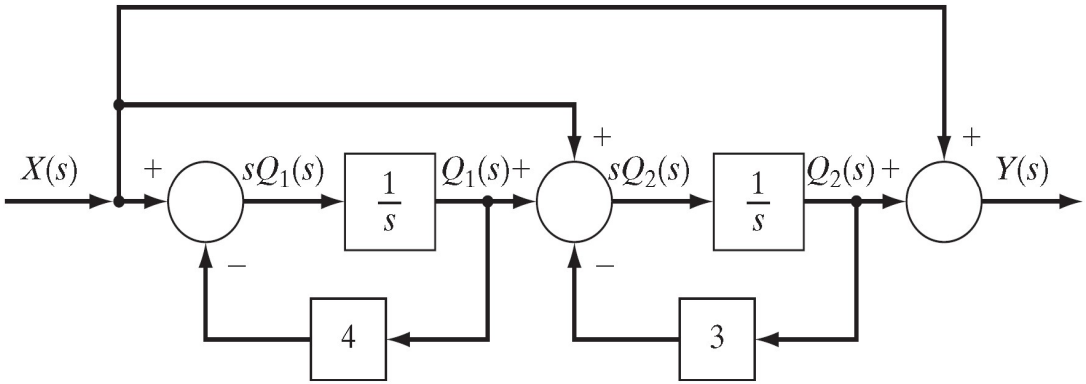




$$sQ_1(s) = -4Q_1(s) + X(s)$$

$$sQ_2(s) = Q_1(s) - 3Q_2(s) + X(s)$$

$$Y(s) = Q_2(s) + X(s)$$



$$Q_1(s) = \frac{1}{s+4}X(s)$$

$$Q_2(s) = \frac{1}{s+3}[Q_1(s) + X(s)] = \frac{1}{s+3}\left(\frac{1}{s+4} + 1\right)X(s) = \frac{s+5}{(s+3)(s+4)}X(s)$$

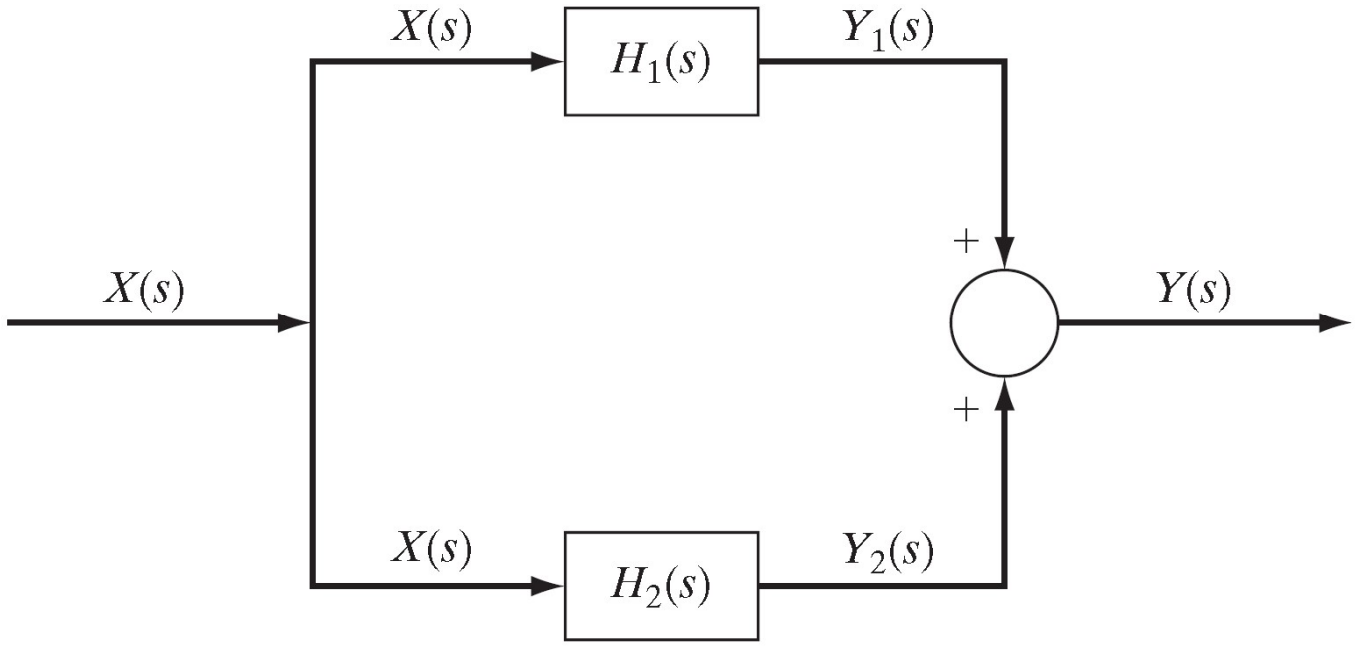
$$Y(s) = \left[\frac{s+5}{(s+3)(s+4)} + 1\right]X(s) = \frac{s^2+8s+17}{(s+3)(s+4)}X(s)$$

$$H(s) = \frac{s^2+8s+17}{(s+3)(s+4)} = \frac{s^2+8s+17}{s^2+7s+12}$$

Transfer Function of Block Diagrams

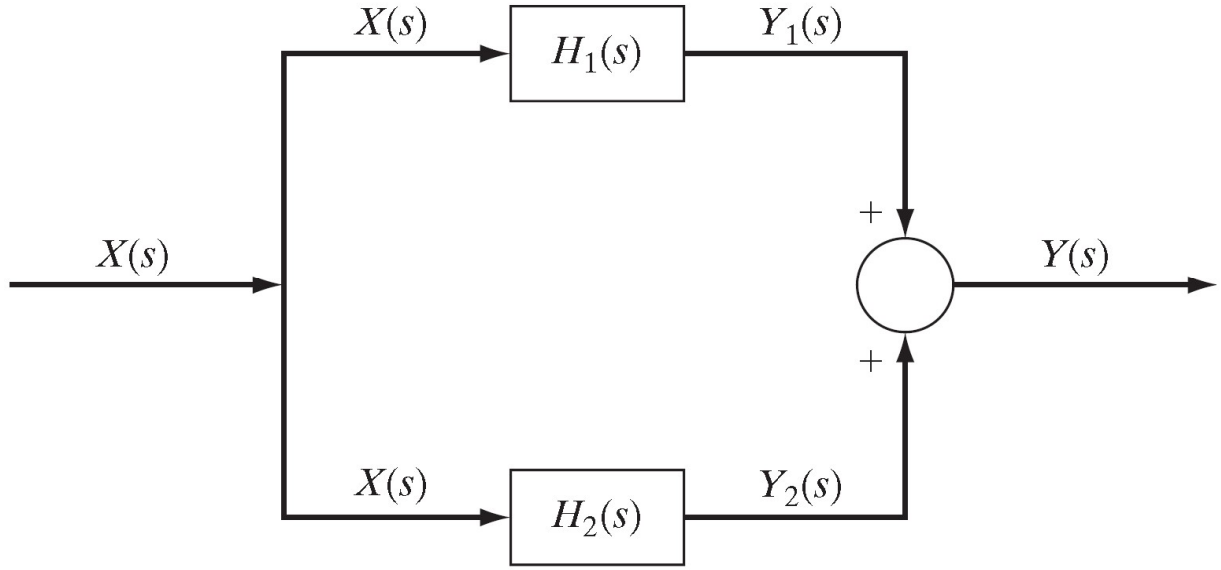
A linear time-invariant continuous-time system is sometimes specified by a block diagram consisting of an interconnection of "blocks," with each block represented by a transfer function.

1- Parallel Interconnection:



$$Y(s) = Y_1(s) + Y_2(s)$$

$$Y(s) = Y_1(s) + Y_2(s)$$

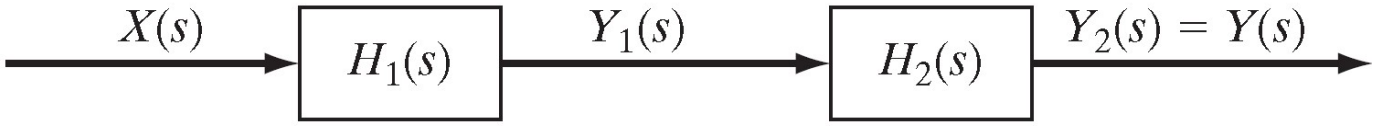


$$\begin{aligned} Y(s) &= H_1(s)X(s) + H_2(s)X(s) \\ &= (H_1(s) + H_2(s))X(s) \end{aligned}$$

Then, for parallel connections the equivalent transfer function is:

$$H(s) = H_1(s) + H_2(s)$$

2- Series Interconnection:



$$Y_1(s) = H_1(s)X(s) \qquad Y_2(s) = H_2(s)Y_1(s)$$

$$Y(s) = Y_2(s) = H_2(s)H_1(s)X(s)$$

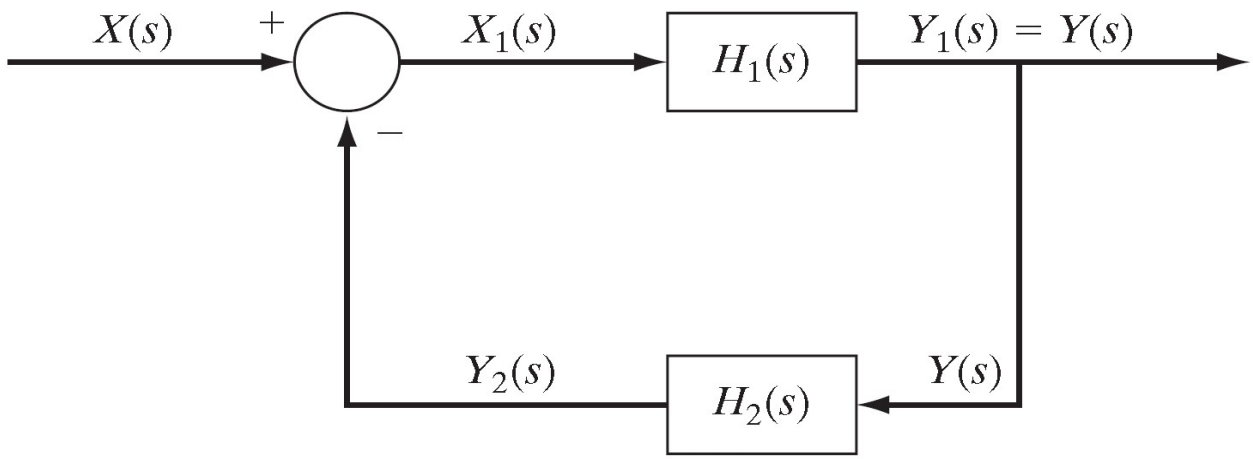
$$H(s) = H_2(s)H_1(s)$$

$$H_2(s)H_1(s) = H_1(s)H_2(s)$$

$$\boxed{H(s) = H_1(s)H_2(s)}$$

3- Feedback Interconnection:

$$Y(s) = H_1(s)X_1(s)$$



$$X_1(s) = X(s) - Y_2(s) = X(s) - H_2(s)Y(s)$$

$$Y(s) = H_1(s)[X(s) - H_2(s)Y(s)]$$

$$[1 + H_1(s)H_2(s)]Y(s) = H_1(s)X(s)$$

$$Y(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}X(s) \quad \Rightarrow \quad \boxed{H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}}$$