EE 362: Signals and Systems

Laplace Transform & Transfer Function Representation – Chapter 6

Lecture 16

Chapter Contents

- Laplace transform of a signal
- Properties of Laplace transform
- Computation of inverse Laplace transform
- Transform of input/output differential equation
- Transform of input/output convolution integral
- Direct construction of the transfer function

Computation of inverse Laplace transform

Given a signal $x(t)$ with Laplace transform $X(s)$, we can compute $x(t)$ from $X(s)$ by taking the inverse Laplace transform of $X(s)$ as follows,

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}\,ds$$

This integral is usually difficult to evaluate. Laplace transform can be calculated by expressing $X(s)$ as a rational function of s as follows,

$$X(s) = \frac{B(s)}{A(s)}$$

Where $B(s)$ and $A(s)$ are polynomials in the complex variable s given by,

$$B(s) = b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0$$

$$A(s) = a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0$$

M and N are positive integers which represent the degrees of the polynomials and the coefficients $b_M, b_{M-1}, \ldots, b_1, b_0$ and $a_N, a_{N-1}, \ldots, a_1, a_0$ are real numbers.

The degree N of the denominator polynomial $A(s)$ is called the ***order*** of the rational function.

- For a large scale of signals $x(t)$, the Laplace transform is rational. An exception is the c-second rectangular pulse $u(t) - u(t - c)$, whose Laplace transform is

$$\frac{1 - e^{-cs}}{s}$$

Due to the presence of the complex exponential e^{-cs}, this transform cannot be expressed as a ratio of polynomials in s, and thus the transform of the rectangular pulse is not a rational function of s.

Given a rational transform $X(s) = B(s)/A(s)$, let $p_1, p_2, \ldots, p_N$ denote the roots of the equation

$$A(s) = 0$$

Then $A(s)$ can be written in the factored form

$$A(s) = a_N(s - p_1)(s - p_2)\cdots(s - p_N)$$

The roots $p_1, p_2, \ldots, p_N$, which may be real or complex, are also said to be the *zeros* of the polynomial $A(s)$ (poles of the system).

If any one of the zeros is complex (say p_1), there must be another zero that is equal to the conjugate of p_1.

TABLA 6.2 Common Laplace Transform Pairs

$$u(t) \leftrightarrow \frac{1}{s}$$

$$u(t) - u(t - c) \leftrightarrow \frac{1 - e^{-cs}}{s}, c > 0$$

$$t^N u(t) \leftrightarrow \frac{N!}{s^{N+1}}, N = 1, 2, 3, \ldots$$

$$\delta(t) \leftrightarrow 1$$

$$\delta(t - c) \leftrightarrow e^{-cs}, c > 0$$

$$e^{-bt} u(t) \leftrightarrow \frac{1}{s + b}, b \text{ real or complex}$$

$$t^N e^{-bt} u(t) \leftrightarrow \frac{N!}{(s + b)^{N+1}}, \quad N = 1, 2, 3, \ldots$$

$$(\cos \omega t)u(t) \leftrightarrow \frac{s}{s^2 + \omega^2}$$

$$(\sin \omega t)u(t) \leftrightarrow \frac{\omega}{s^2 + \omega^2}$$

$$(\cos^2 \omega t)u(t) \leftrightarrow \frac{s^2 + 2\omega^2}{s(s^2 + 4\omega^2)}$$

$$(\sin^2 \omega t)u(t) \leftrightarrow \frac{2\omega^2}{s(s^2 + 4\omega^2)}$$

$$(e^{-bt} \cos \omega t)u(t) \leftrightarrow \frac{s + b}{(s + b)^2 + \omega^2}$$

$$(e^{-bt} \sin \omega t)u(t) \leftrightarrow \frac{\omega}{(s + b)^2 + \omega^2}$$

$$(t\cos\omega t)u(t) \leftrightarrow \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

$$(t\sin\omega t)u(t) \leftrightarrow \frac{2\omega s}{(s^2+\omega^2)^2}$$

$$(te^{-bt}\cos\omega t)u(t) \leftrightarrow \frac{(s+b)^2-\omega^2}{[(s+b)^2+\omega^2]^2}$$

$$(te^{-bt}\sin\omega t)u(t) \leftrightarrow \frac{2\omega(s+b)}{[(s+b)^2+\omega^2]^2}$$

- MATLAB can be used to find the zeros of a polynomial by the command "roots"
- Examples: find the zeros of

$$A(s) = s^3 + 4s^2 + 6s + 4 = 0$$

```
A=[1  4  6  4];
p=roots(A)
```

```
p =
   -2.0000 + 0.0000i
   -1.0000 + 1.0000i
   -1.0000 - 1.0000i
```

Hence,

$$A(s) = (s + 2)(s + 1 - j)(s + 1 + j)$$

Then the rational transform $X(s)$ is

$$X(s) = \frac{B(s)}{a_N(s - p_1)(s - p_2)\cdots(s - p_N)}$$

The p_i for $i = 1, 2, \ldots, N$ are called the **poles** of the rational $X(s)$.

- If $M < N$, the rational function is said to be strictly proper in s. For this category we will study (1) distinct real poles, (2) distinct real and complex poles, and (3) repeated poles
- If the rational function is not strictly proper in s, we will divide it into a polynomial and a rational function.

- **Distinct Poles**: $p_i \neq p_j$ when $i \neq j$

$$X(s) = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \cdots + \frac{c_N}{s - p_N}$$

where

$$c_i = [(s - p_i)X(s)]_{s=p_i}, \quad i = 1, 2, \ldots, N$$

The constants c_i are called the ***residues***, and the computation of the c_i using the above equation is called the ***residue method***.

We can determine inverse Laplace transform $x(t)$ of $X(s)$ by taking the inverse transform of each term as follows,

$$x(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t} + \cdots + c_N e^{p_N t}, \quad t \geq 0$$

- MATLAB can be used to compute the residues and the poles of $X(s)$.
- Example:

$$X(s) = \frac{s + 2}{s^3 + 4s^2 + 3s}$$

```
num=[1 2];
den=[1 4 3 0];
[r,p]=residue(num,den)
```

```
r =
   -0.1667
   -0.5000
    0.6667
p =
   -3
   -1
    0
```

Example: suppose that

$$X(s) = \frac{s+2}{s^3 + 4s^2 + 3s}$$

Here

$$A(s) = s^3 + 4s^2 + 3s = s(s+1)(s+3)$$

The roots of $A(s) = 0$ are $0, -1, -3$, and thus the poles of $X(s)$ are $p_1 = 0, p_2 = -1, p_3 = -3$. Then

$$X(s) = \frac{c_1}{s-0} + \frac{c_2}{s-(-1)} + \frac{c_3}{s-(-3)}$$

$$X(s) = \frac{c_1}{s} + \frac{c_2}{s+1} + \frac{c_3}{s+3}$$

where

$$c_1 = [sX(s)]_{s=0} = \left.\frac{s+2}{(s+1)(s+3)}\right|_{s=0} = \frac{2}{3}$$

$$c_2 = [(s+1)X(s)]_{s=-1} = \left.\frac{s+2}{s(s+3)}\right|_{s=-1} = \frac{1}{-2}$$

$$c_3 = [(s+3)X(s)]_{s=-3} = \left.\frac{s+2}{s(s+1)}\right|_{s=-3} = \frac{-1}{6}$$

Hence the inverse Laplace transform is

$$x(t) = \frac{2}{3} - \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t}, \quad t \geq 0$$

Distinct Poles with Two or More Complex Poles:

Suppose that $p_1 = \sigma + j\omega$ and $\omega \neq 0$. Then, $\bar{p}_1 = \sigma - j\omega$ is another pole of $X(s)$. The residue $\bar{c}_1$ that corresponds to $\bar{p}_1$ is the complex conjugate of c_1 that corresponds to p_1. Then, $X(s)$ can be expressed as

$$X(s) = \frac{c_1}{s - p_1} + \frac{\bar{c}_1}{s - \bar{p}_1} + \frac{c_3}{s - p_3} + \cdots + \frac{c_N}{s - p_N}$$

where $c_1, c_2, \ldots, a_N$ are calculated as before.

The inverse Laplace transform is

$$x(t) = c_1 e^{p_1 t} + \bar{c}_1 e^{\bar{p}_1 t} + c_3 e^{p_3 t} + \cdots + c_N e^{p_N t}$$

The first two terms can be expressed as,

$$c_1 e^{p_1 t} + \overline{c}_1 e^{\overline{p}_1 t} = 2|c_1| e^{\sigma t} \cos(\omega t + \angle c_1)$$

Then, the inverse of Laplace transform is,

$$x(t) = 2|c_1| e^{\sigma t} \cos(\omega t + \angle c_1) + c_3 e^{p_3 t} + \cdots + c_N e^{p_N t}$$

There are two other methods to avoid working with complex poles: completing the square and equating coefficients (please refer to the textbook)

Example: Suppose that $X(s) = \dfrac{s^2 - 2s + 1}{s^3 + 3s^2 + 4s + 2}$

Here, $A(s) = s^3 + 3s^2 + 4s + 2 = (s + 1 - j)(s + 1 + j)(s + 1)$

The roots of $A(s) = 0$ are

$$p_1 = -1 + j, \qquad p_2 = -1 - j, \qquad p_3 = -1$$

Thus, $\sigma = -1$ and $\omega = 1$, and

$$X(s) = \frac{c_1}{s - (-1 + j)} + \frac{\bar{c}_1}{s - (-1 - j)} + \frac{c_3}{s - (-1)}$$

$$X(s) = \frac{c_1}{s + 1 - j} + \frac{\bar{c}_1}{s + 1 + j} + \frac{c_3}{s + 1}$$

where

$$c_1 = [(s + 1 - j)X(s)]_{s=-1+j} = \left.\frac{s^2 - 2s + 1}{(s + 1 + j)(s + 1)}\right|_{s=-1+j} = \frac{-3}{2} + j2$$

$$c_3 = [(s + 1)X(s)]_{s=-1} = \left.\frac{s^2 - 2s + 1}{s^2 + 2s + 2}\right|_{s=-1} = 4$$

Then,

$$|c_1| = \sqrt{\frac{9}{4} + 4} = \frac{5}{2}$$

Since the real part of c_1 is negative, c_1 lines in the second quadrant, then

$$\angle c_1 = 180^\circ + \tan^{-1}\frac{-4}{3} = 126.87^\circ$$

Then

$$x(t) = 5e^{-t}\cos(t + 126.87^\circ) + 4e^{-t}, \qquad t \geq 0$$

Using MATLAB:

```
num=[1 -2 1];
den=[1 3 4 2];
[r,p]=residue(num,den)
```

```
r =
  -1.5000 + 2.0000i
  -1.5000 - 2.0000i
   4.0000 + 0.0000i
p =
  -1.0000 + 1.0000i
  -1.0000 - 1.0000i
  -1.0000 + 0.0000i
```

Repeated Poles: Consider the general case,

$$X(s) = \frac{B(s)}{A(s)}$$

Suppose that pole p_1 is repeated r times and the other $N - r$ poles (denoted by $p_{r+1}, p_{r+2}, \ldots, p_N$) are distinct.

Then, $X(s)$ has the partial fraction expansion

$$X(s) = \frac{c_1}{s - p_1} + \frac{c_2}{(s - p_1)^2} + \cdots + \frac{c_r}{(s - p_1)^r}$$

$$+ \frac{c_{r+1}}{s - p_{r+1}} + \cdots + \frac{c_N}{s - p_N}$$

The residues $c_{r+1}, c_{r+2}, \ldots, c_N$ are calculated as before,

$$c_i = [(s - p_i)X(s)]_{s=p_i}, \quad i = r+1, r+2, \ldots, N$$

The constant c_r is given by

$$c_r = [(s - p_1)^r X(s)]_{s=p_1}$$

and the constants $c_1, c_2, \ldots, c_{r-1}$ are given by

$$c_{r-i} = \frac{1}{i!}\left[\frac{d^i}{ds^i}[(s - p_1)^r X(s)]\right]_{s=p_1}, \quad i = 1, 2, \ldots, r-1$$

For example, if the index i of the previous equation is considered equal to 1, 2. Then,

$$c_{r-2} = \frac{1}{2}\left[\frac{d^2}{ds^2}[(s - p_1)^r X(s)]\right]_{s=p_1} \quad c_{r-1} = \left[\frac{d}{ds}[(s - p_1)^r X(s)]\right]_{s=p_1}$$

If the poles of $X(s)$ are all real numbers, the inverse Laplace transform can be determined from transform pairs as follows,

$$\frac{t^{N-1}}{(N-1)!}e^{-at} \leftrightarrow \frac{1}{(s+a)^N}, \quad N = 1, 2, 3, \ldots$$

If $X(s)$ has repeated complex poles, the complex part of $A(s)$ can be expressed in quadratic terms.

Example: Repeated poles: Consider the rational function

$$X(s) = \frac{5s - 1}{s^3 - 3s - 2}$$

The roots of $A(s) = 0$ are $-1, -1, 2$, so $r = 2$, and therefore the partial fraction expansion has the form,

$$X(s) = \frac{c_1}{s + 1} + \frac{c_2}{(s + 1)^2} + \frac{c_3}{s - 2}$$

where

$$c_1 = \left[\frac{d}{ds}[(s + 1)^2 X(s)]\right]_{s=-1} = \left[\frac{d}{ds}\frac{5s - 1}{s - 2}\right]_{s=-1} = \left.\frac{-9}{(s - 2)^2}\right|_{s=-1} = -1$$

$$c_2 = [(s+1)^2 X(s)]_{s=-1} = \left.\frac{5s-1}{s-2}\right|_{s=-1} = 2$$

$$c_3 = [(s-2)X(s)]_{s=2} = \left.\frac{5s-1}{(s+1)^2}\right|_{s=2} = 1$$

Thus,

$$x(t) = -e^{-t} + 2te^{-t} + e^{2t}, \quad t \geq 0$$

Instead of having to differentiate with respect to s, we can compute the constant c_1 by putting the partial fraction expansion

$$X(s) = \frac{c_1}{s+1} + \frac{2}{(s+1)^2} + \frac{1}{s-2}$$

over a common denominator and then equating numerators.

$$5s - 1 = c_1(s + 1)(s - 2) + 2(s - 2) + (s + 1)^2$$

$$5s - 1 = (c_1 + 1)s^2 + (-c_1 + 4)s + (-2c_1 - 3)$$

Hence, $c_1 = -1$ as before.

Using MATLAB

```
num=[5 -1];
den=[1 0 -3 -2];
[r,p]=residue(num,den)
```

```
r =
    1.0000
   -1.0000
    2.0000
p =
    2.0000
   -1.0000
   -1.0000
```

Case when $M \geq N$: Consider the rational function $X(s) = B(s)/A(s)$, with the degree of $B(s)$ equal to M and the degree of $A(s)$ equal to N. If $M \geq N$, by long division, $X(s)$ can be rewritten in the form

$$X(s) = Q(s) + \frac{R(s)}{A(s)}$$

where $Q(s)$ is a polynomial in s with degree strictly less than N.

MATLAB can be used to solve for $Q(s)$ and $R(s)$.

Example: $M = 3$ and $N = 2$

Suppose that

$$X(s) = \frac{s^3 + 2s - 4}{s^2 + 4s - 2}$$

Using MATLAB commands

```
num=[1 0 2 -4];
den=[1 4 -2];
[Q,R]=deconv(num,den)
```

gives `Q=[1 -4]` and `R=[20 -12]`. Thus, the quotient is $Q(s) = s - 4$, and the remainder is $R(s) = 20s - 12$. Then,

$$X(s) = s - 4 + \frac{20s - 12}{s^2 + 4s - 2}$$

and thus

$$x(t) = \frac{d}{dt}\delta(t) - 4\delta(t) + v(t)$$

where $v(t)$ is the inverse Laplace transform of

$$V(s) = \frac{20s - 12}{s^2 + 4s - 2}$$

Using MATLAB

```
num=[20 -12];
den=[1 4 -2];
[r,p]=residue(num,den)
```

gives

```
r =
   20.6145
   -0.6145
p =
   -4.4495
    0.4495
```

Hence,

$$V(s) = \frac{20.6145}{s + 4.4495} - \frac{0.6145}{s - 0.4495}$$

and

$$v(t) = 20.6145e^{-4.4495t} - 0.6145e^{0.4495t}, t \geq 0$$

Inverse Laplace computation using MATLAB

The command is “ilaplace”

Example: suppose that

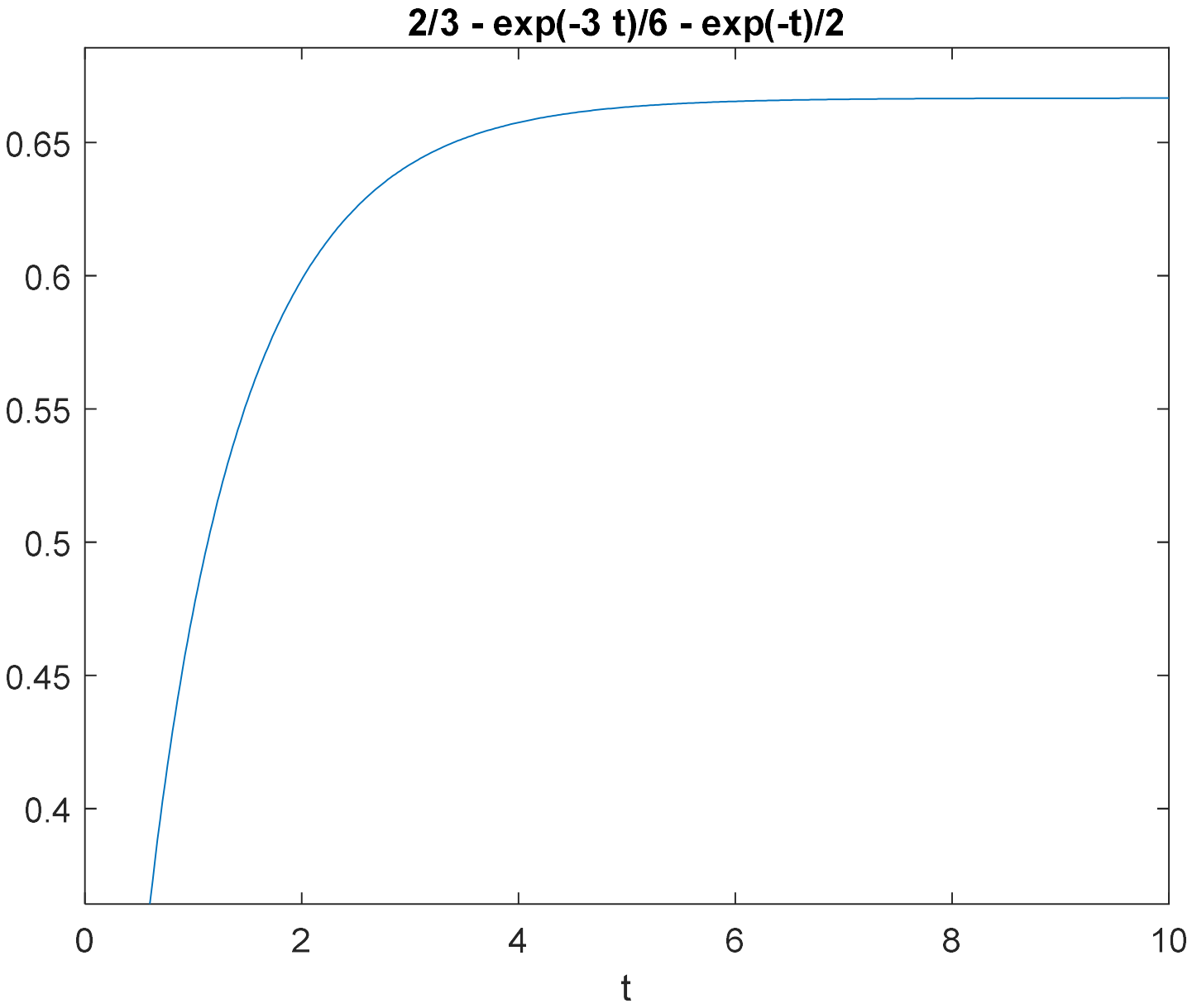
$$X(s) = \frac{s+2}{s^3 + 4s^2 + 3s}$$

```
syms X s x
X=(s+2)/(s^3+4*s^2+3*s);
x=ilaplace(X)
```

```
x =
2/3 - exp(-3*t)/6 - exp(-t)/2
```

To plot the function:

```
ezplot(x,[0,10])
```


2/3 - exp(-3 t)/6 - exp(-t)/2
0.65
0.6
0.55
0.5
0.45
0.4
0
2
4
6
8
10
t

Example: Repeated complex poles

Suppose that $$X(s) = \frac{s^3 + 3s^2 - s + 1}{s^5 + s^4 + 2s^3 + 2s^2 + s + 1}$$

Using MATLAB command "roots" reveals that the poles of $X(s)$ are equal to $-1, -j, -j, j, j$. Thus, there are a pair of repeated complex poles corresponding to the factor

$$[(s + j)(s - j)]^2 = (s^2 + 1)^2$$

Therefore,

$$X(s) = \frac{s^3 + 3s^2 - s + 1}{(s^2 + 1)^2(s + 1)}$$

and the expansion of $X(s)$ has the form,

$$X(s) = \frac{cs + d}{s^2 + 1} + \frac{w(s)}{(s^2 + 1)^2} + \frac{c_5}{s + 1}$$

where $\omega(s)$ is a polynomial in s. To be able to use table 6.2, use the following

$$\frac{w(s)}{(s^2 + 1)^2} = \frac{g(s^2 - 1) + hs}{(s^2 + 1)^2}$$

for some constants g and h. From table 6.2,

$$\left(gt \cos t + \frac{h}{2} t \sin t \right) u(t) \leftrightarrow \frac{g(s^2 - 1) + hs}{(s^2 + 1)^2}$$

Now,

$$c_5 = [(s+1)X(s)]_{s=-1} = \left.\frac{s^3 + 3s^2 - s + 1}{(s^2+1)^2}\right|_{s=-1} = 1$$

So,

$$X(s) = \frac{cs + d}{s^2 + 1} + \frac{g(s^2 - 1) + hs}{(s^2 + 1)^2} + \frac{1}{s + 1}$$

Putting the right-hand side over a common denominator and equating numerators give

$$s^3 + 3s^2 - s + 1 = (cs + d)(s^2 + 1)(s + 1) + [g(s^2 - 1) + hs](s + 1) + (s^2 + 1)^2$$

$$s^3 + 3s^2 - s + 1 = (c + 1)s^4 + (c + d + g)s^3 + (c + d + h + g + 2)s^2 + (c + d - g + h)s + d - g + 1$$

Equating coefficients of the polynomials gives,

$$c = -1$$

$$c + d + g = 1$$

$$c + d + h + g + 2 = 3$$

$$c + d - g + h = -1$$

$$d - g + 1 = 1$$

Solving these equations gives $d = 1$, $g = 1$, and $h = 0$, and thus the inverse Laplace transform is

$$x(t) = -\cos t + \sin t + t\cos t + e^{-t}, t \geq 0$$