EE 362: Signals and Systems

Laplace Transform & Transfer Function Representation – Chapter 6

Lecture 15

Chapter Contents

- Laplace transform of a signal
- Properties of Laplace transform
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Laplace transform of a signal

Given a continuous-time signal $x(t)$, the Fourier transform $X(\omega)$ of $x(t)$ is

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Some signals do not have Fourier transform in the ordinary sense such as unit-step function. The Fourier transform of the unit-step can be expressed as follows,

$$X(\omega) = \int_{0}^{\infty} e^{-j\omega t}\,dt$$

This integral does not exist. However, if an exponential convergence factor $e^{-\sigma t}$ is added, this problem can be circumvented where σ is a real number. Then,

$$X(\omega) = \int_0^\infty e^{-\sigma t} e^{-j\omega t}\, dt$$

$$X(\omega) = \int_0^\infty e^{-(\sigma + j\omega)t}\, dt$$

Now, $X(\omega)$ is a function in a complex number

$$X(\sigma + j\omega) = \int_0^\infty e^{-(\sigma + j\omega)t}\, dt$$

$$X(\sigma + j\omega) = -\frac{1}{\sigma + j\omega}[e^{-(\sigma + j\omega)t}]_{t=0}^{t=\infty}$$

Now, $\lim_{t\to\infty} e^{-(\sigma+j\omega)t}$ exists if and only if $\sigma>0$, in which case

$$\lim_{t\to\infty} e^{-(\sigma+j\omega)t} = 0$$

Then,

$$X(\sigma + j\omega) = -\frac{1}{\sigma + j\omega}[0 - e^{-(\sigma+j\omega)(0)}] = \frac{1}{\sigma + j\omega}$$

By letting s denote the complex number $\sigma + j\omega$,

$$X(s) = \frac{1}{s}$$

This is the Laplace transform of the unit-step function. Note that $X(s)$ is a complex-valued function of the complex number s.

This will be defined only for those complex numbers s for which the real part of s (which is equal to σ) is strictly positive. In this particular example, the integral does not exist for $\sigma \leq 0$. In other words, the ***region of convergence*** of Laplace transform of the unit-step function is

$$\text{Re}\, s = \sigma > 0$$

This concept can be generalized for a large class of signals $x(t)$ as follows

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t}\, dt$$

Or,

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

This transformation is called ***two-sided*** (or bilateral) Laplace transformation.

The ***one-sided*** (or unilateral) Laplace transform of $x(t)$, also denoted $X(s)$, is defined by

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}\,dt$$

The value of the signal for $t < 0$ is treated as initial condition. Our analysis will be limited to one-sided Laplace transform

Example: determine Laplace transform and its region of convergence of the following signal

$$x(t) = e^{-bt}u(t)$$

$$X(s) = \int_0^\infty e^{-bt}e^{-st}\,dt = \int_0^\infty e^{-(s+b)t}\,dt$$

$$X(s) = -\frac{1}{s+b}\left[e^{-(s+b)t}\right]_{t=0}^{t=\infty}$$

Setting $s = \sigma + j\omega$ will result in

$$\lim_{t\to\infty} e^{-(\sigma+j\omega+b)t}$$

This limit will exist only if $\sigma + b > 0$, in which case the limit is zero.

Then,

$$X(s) = \frac{1}{s + b}$$

and the region of convergence is $Re\ s > -b$

- The two sided Laplace transform can be viewed as a generalization of the definition of the Fourier transform.

$$X(\omega) = X(s)\big|_{s=j\omega}$$

This is valid if and only if the region of convergence for $X(s)$ includes $\sigma = 0$. For example, for a unit-step function, this relationship is not valid since the region of convergence is $Re\ s > 0$, which does not include the point $\mathrm{Re\ s} = \sigma = 0$.

Example: If $x(t) = e^{-bt}u(t)$, (b is an arbitrary real number) the Laplace transform is $X(s) = 1/(s+b)$ and the region of convergence is $Re\ s > -b$.

If $b > 0$, the region of convergence includes $\sigma = 0$, and the Fourier transform $X(\omega)$ of $x(t)$ is given by

$$X(\omega) = X(s)\big|_{s=j\omega} = \frac{1}{j\omega + b}$$

If $b \geq 0$, $x(t)$ does not have a Fourier transform in the ordinary sense but it does have a Laplace transform

$$X(s) = \frac{1}{s + b}$$

- Transform pair notation $x(t) \leftrightarrow X(s)$

e.g., $$e^{-bt}u(t) \leftrightarrow \frac{1}{s+b}$$

Some other notations:

$$X(s) = L[x(t)]$$

$$x(t) = L^{-1}[X(s)]$$

- **<u>Laplace Transform Computation Using Symbolic Manipulation</u>**

laplace command in MATLAB, X=laplace(x).

Example:

```
syms x b t
x=exp(-b*t);
X=laplace(x)
```

```
Gives: X =1/(b + s)
```

Example: Laplace transform of a unit-step function

```
x=sym(1);
X=laplace(x)
```

```
Gives: X =1/s
```

where sym(1)is used to create a symbolic object of the number 1.

Properties of the Laplace Transform

1- Linearity: Laplace transform is a linear operation.

If $x(t) \leftrightarrow X(s)$ and $v(t) \leftrightarrow V(s)$, then

$$ax(t) + bv(t) \leftrightarrow aX(s) + bV(s)$$

Example: Consider the signal $x(t) = u(t) + e^{-t}u(t)$. Then, using the linearity property,

$$u(t) + e^{-t}u(t) \leftrightarrow \frac{1}{s} + \frac{1}{s+1} = \frac{2s+1}{s(s+1)}$$

2- Right shift in time: if $x(t) \leftrightarrow X(s)$, then for any positive real number c,

$$x(t-c)u(t-c) \leftrightarrow e^{-cs}X(s)$$

Example: Let

$$x(t) = \begin{cases} 1, & 0 \le t < c \\ 0, & \text{all other t} \end{cases}$$

where c is a positive real number. $x(t)$ can be expressed as, $x(t) = u(t) - u(t-c)$. Then,

$$u(t) - u(t-c) \leftrightarrow \frac{1}{s} - \frac{e^{-cs}}{s} = \frac{1-e^{-cs}}{s}$$

3- Time scaling: if $x(t) \leftrightarrow X(s)$, for any positive real number a,

$$x(at) \leftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right)$$

Example: Consider the time-scaled unit-step function $u(at)$, where a is an arbitrary positive real number.

$$u(at) \leftrightarrow \frac{1}{a}\left(\frac{1}{s/a}\right) = \frac{1}{s}$$

This result is not unexpected since $u(at) = u(t)$ for any real number $a > 0$

4- Multiplication by a power of t: if $x(t) \leftrightarrow X(s)$, then for any positive integer N,

$$t^N x(t) \leftrightarrow (-1)^N \frac{d^N}{ds^N} X(s)$$

Example: Consider the unit-ramp function $r(t) = tu(t)$. Then,

$$R(s) = -\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2}$$

For $x(t) = t^N u(t)$, $X(s)$ is

$$t^N u(t) \leftrightarrow \frac{N!}{s^{N+1}}$$

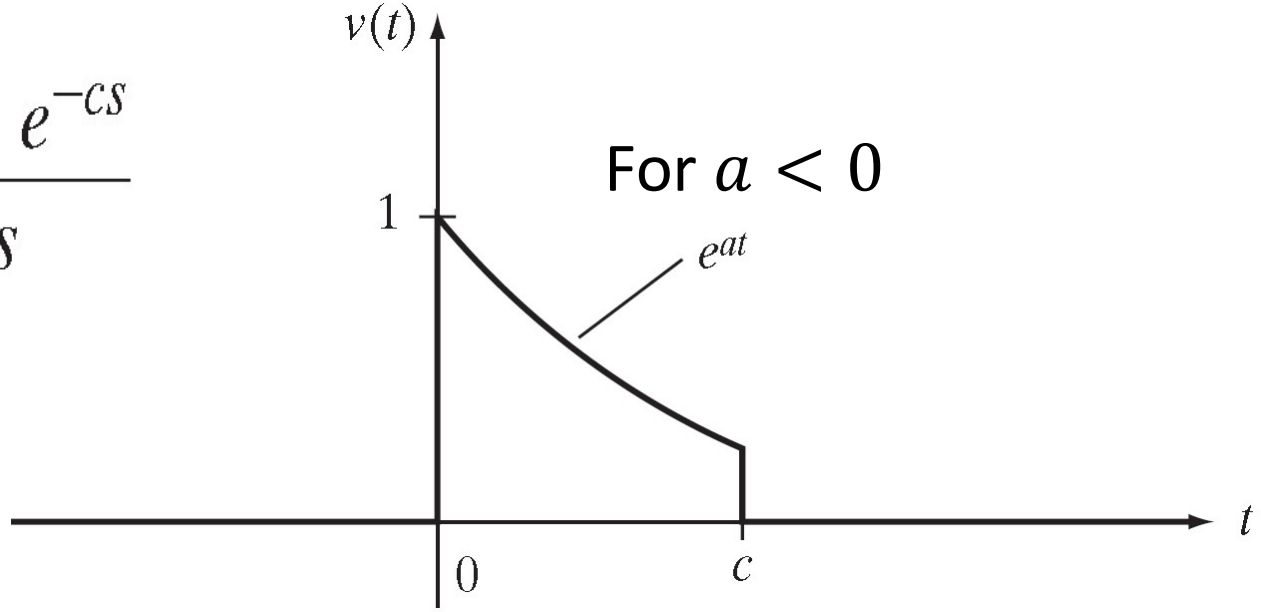
5- Multiplication by an Exponential: if $x(t) \leftrightarrow X(s)$, then for any real or complex number a,

$$e^{at}x(t) \leftrightarrow X(s - a)$$

Example: Let $v(t) = [u(t) - u(t - c)]e^{at}$, where c is a positive real number and a is any real number.

$$u(t) - u(t - c) \leftrightarrow \frac{1 - e^{-cs}}{s}$$

$$V(s) = \frac{1 - e^{-c(s-a)}}{s - a}$$



6- Multiplication by a Sinusoid: if $x(t) \leftrightarrow X(s)$, then for any real number ω,

$$x(t)\sin \omega t \leftrightarrow \frac{j}{2}[X(s + j\omega) - X(s - j\omega)]$$

$$x(t)\cos \omega t \leftrightarrow \frac{1}{2}[X(s + j\omega) + X(s - j\omega)]$$

This can be approved by

$$x(t)\sin \omega t = \frac{j}{2}x(t)[e^{-j\omega t} - e^{j\omega t}]$$

$$x(t)\cos \omega t = \frac{1}{2}x(t)[e^{-j\omega t} + e^{j\omega t}]$$

$$e^{\pm j\omega t}x(t) \leftrightarrow X(s \mp j\omega)$$

Example: Let $v(t) = (\cos(\omega t))u(t)$. We know that $u(t) \leftrightarrow 1/s$, then

$$V(s) = \frac{1}{2}\left(\frac{1}{s + j\omega} + \frac{1}{s - j\omega}\right) = \frac{1}{2}\frac{s - j\omega + s + j\omega}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2}$$

Then

$$(\cos \omega t)u(t) \leftrightarrow \frac{s}{s^2 + \omega^2}$$

$$(\sin \omega t)u(t) \leftrightarrow \frac{\omega}{s^2 + \omega^2}$$

7- Differentiation in Time Domain: if $x(t) \leftrightarrow X(s)$, then,

$$\dot{x}(t) \leftrightarrow sX(s) - x(0)$$

Proof: $\int_0^\infty \dot{x}(t)e^{-st}\,dt$

$$\int_0^\infty \dot{x}(t)e^{-st}\,dt = vw|_{t=0}^{t=\infty} - \int_0^\infty w\,dv$$

$$= e^{-st}x(t)|_{t=0}^{t=\infty} - \int_0^\infty x(t)(-s)e^{-st}\,dt = \lim_{t\to\infty}[e^{-st}x(t)] - x(0) + sX(s)$$

$$\lim_{t\to\infty} e^{-st}x(t) = 0$$

$$\int_0^\infty \dot{x}(t)e^{-st}\,dt = -x(0) + sX(s)$$

$\dot{x}(t) \leftrightarrow sX(s) - x(0^-)$ If $x(t) = 0$ for t<0 $\dot{x}(t) \leftrightarrow sX(s)$

Example: Let $x(t) = u(t)$. Then, $\dot{x}(t) = \delta(t)$

$$\dot{x}(t) \leftrightarrow s\frac{1}{s} - u(0^-) = 1 - 0 = 1$$

$$\delta(t) \leftrightarrow 1$$

Second and higher derivatives

$$\frac{d^2x(t)}{dt^2} \leftrightarrow s^2X(s) - sx(0) - \dot{x}(0)$$

$$x^{(N)}(t) \leftrightarrow s^N X(s) - s^{N-1}x(0) - s^{N-2}\dot{x}(0) - \cdots - sx^{(N-2)}(0) - x^{(N-1)}(0)$$

8- Integration: if $x(t) \leftrightarrow X(s)$, then,

$$\int_0^t x(\lambda)\, d\lambda \leftrightarrow \frac{1}{s} X(s)$$

Example: Let $x(t) = u(t)$. The integral of $u(t)$ is the unit-ramp function, $r(t) = tu(t)$

$$u(t) \leftrightarrow \frac{1}{s}$$

$$r(t) = \int_0^t u(\lambda) d\lambda \leftrightarrow \frac{1}{s}\frac{1}{s}$$

$$r(t) \leftrightarrow \frac{1}{s^2}$$

9- Convolution: given two signals $x(t)$ and $v(t)$ with both $x(t)$ and $v(t)$ equal to zero for all $t < 0$.

$$x(t) * v(t) \leftrightarrow X(s)V(s)$$

Example: Let $x(t) = u(t) - u(t-1)$. Find Laplace transform of $x(t) * x(t)$

$$u(t) - u(t-c) \leftrightarrow \frac{1 - e^{-cs}}{s}$$

$$X^2(s) = \left(\frac{1 - e^{-s}}{s}\right)^2 = \frac{1 - 2e^{-s} + e^{-2s}}{s^2}$$

10- Initial-Value Theorem: Given a signal $x(t)$ with a transform $X(s)$, the initial values $x(0)$ and $\dot{x}(0)$ can be computed as follows:

$$x(0) = \lim_{s\to\infty} sX(s)$$

$$\dot{x}(0) = \lim_{s\to\infty} [s^2X(s) - sx(0)]$$

In general:

$$x^{(N)}(0) = \lim_{s\to\infty} [s^{N+1}X(s) - s^Nx(0) - s^{N-1}\dot{x}(0) - \cdots - sx^{(N-1)}(0)]$$

Proof:

$$x(t) = \sum_{i=0}^{\infty} x^{(i)}(0)\frac{t^i}{i!}$$

$$x(t) = \sum_{i=0}^{\infty} x^{(i)}(0)\frac{t^i}{i!}$$

$$sX(s) = \int_0^\infty \sum_{i=0}^{\infty} sx^{(i)}(0)\frac{t^i}{i!}e^{-st}\,dt \qquad sX(s) = \sum_{i=0}^{\infty} x^{(i)}(0)\frac{s}{s^{i+1}} = \sum_{i=0}^{\infty} x^{(i)}(0)\frac{1}{s^i}$$

$$\lim_{s\to\infty} sX(s) = x(0)$$

Example: If

$$X(s) = \frac{-3s^2 + 2}{s^3 + s^2 + 3s + 2}$$

Then

$$\lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{-3s^3 + 2s}{s^3 + s^2 + 3s + 2} = \frac{-3}{1}$$

Thus

$$x(0) = -3$$

11- Final-Value Theorem: Given the signal $x(t)$ with a transform $X(s)$, suppose that $x(t)$ has a limit $x(\infty)$ as $t \to \infty$; that is,

$$x(\infty) = \lim_{t \to \infty} x(t)$$

The existence of the limit $x(\infty)$ turns out to be equivalent to requiring that the region of convergence of $sX(s)$ includes the value $s = 0$.

If $x(\infty)$ exists, the final value theorem states that

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

- The final value theorem is a very useful property, since the limit as $t \to \infty$ of a time signal $x(t)$ can be computed directly from the Laplace transform $X(s)$. However, care must be taken since the limit of $sX(s)$ as $s \to 0$ may exist even thought $x(t)$ does not have a limit as $t \to \infty$.
- Example: suppose that $$X(s) = \frac{1}{s^2 + 1}$$

Then $$\lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{s}{s^2 + 1} = 0$$

But $x(t) = \sin t$, and $\sin t$ does not have a limit as $t \to \infty$. We will elaborate more on this in the next section.

TABLE 6.1 Properties of the Laplace Transform

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(s) + bV(s)$
Right shift in time	$x(t-c)u(t-c) \leftrightarrow e^{-cs}X(s), c > 0$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X\left(\frac{s}{a}\right), a > 0$
Multiplication by a power of t	$t^N x(t) \leftrightarrow (-1)^N \frac{d^N}{ds^N}X(s), N = 1, 2, \ldots$
Multiplication by an exponential	$e^{at}x(t) \leftrightarrow X(s-a)$, a real or complex
Multiplication by $\sin \omega t$	$x(t) \sin \omega t \leftrightarrow \frac{j}{2}[X(s+j\omega) - X(s-j\omega)]$
Multiplication by $\cos \omega t$	$x(t) \cos \omega t \leftrightarrow \frac{1}{2}[X(s+j\omega) + X(s-j\omega)]$
Differentiation in the time domain	$\dot{x}(t) \leftrightarrow sX(s) - x(0)$
Second derivative	$\ddot{x}(t) \leftrightarrow s^2X(s) - sx(0) - \dot{x}(0)$
Nth derivative	$x^{(N)}(t) \leftrightarrow s^N X(s) - s^{N-1}x(0) - s^{N-2}\dot{x}(0) - \cdots - sx^{(N-2)}(0) - x^{(N-1)}(0)$
Integration	$\int_0^t x(\lambda)\, d\lambda \leftrightarrow \frac{1}{s}X(s)$
Convolution	$x(t) * v(t) \leftrightarrow X(s)V(s)$
Initial-value theorem	$x(0) = \lim_{s\to\infty} sX(s)$
	$\dot{x}(0) = \lim_{s\to\infty}[s^2X(s) - sx(0)]$
	$x^{(N)}(0) = \lim_{s\to\infty}[s^{N+1}X(s) - s^N x(0) - s^{N-1}\dot{x}(0) - \cdots - sx^{(N-1)}(0)]$
Final-value theorem	If $\lim_{t\to\infty} x(t)$ exists, then $\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

Example: Let $v(t) = te^{-bt}u(t)$, where b is any positive real number. The Laplace transform of $e^{-bt}u(t)$ is

$$e^{-bt}u(t) \leftrightarrow \frac{1}{s+b}$$

Then,

$$V(s) = -\frac{d}{ds}\frac{1}{s+b} = \frac{1}{(s+b)^2}$$

For $t^N e^{-bt}u(t)$,

$$t^N e^{-bt}u(t) \leftrightarrow \frac{N!}{(s+b)^{N+1}}$$

Example: Let $v(t) = \left(e^{-bt}\cos \omega t\right)u(t)$. We can use $x(t) = e^{-bt}u(t)$ and apply the multiplication by $\cos \omega t$ property or we can use $x(t) = (\cos \omega t)u(t)$ and apply the multiplication by an exponential property. Let's apply the second option

$$(e^{-bt}\cos \omega t)u(t) \leftrightarrow \frac{s+b}{(s+b)^2+\omega^2}$$

Similar approach can be applied on the sine function

$$(e^{-bt}\sin \omega t)u(t) \leftrightarrow \frac{\omega}{(s+b)^2+\omega^2}$$

Example: Let $v(t) = \left(\sin^2 \omega t\right) u(t)$. Set $x(t) = (\sin \omega t) u(t)$ and use the multiplication by sine property

$$V(s) = \frac{j}{2}\left[\frac{\omega}{(s + j\omega)^2 + \omega^2} - \frac{\omega}{(s - j\omega)^2 + \omega^2}\right]$$

$$= \frac{j}{2}\frac{\omega(s - j\omega)^2 + \omega^3 - \omega(s + j\omega)^2 - \omega^3}{(s + j\omega)^2(s - j\omega)^2 + \omega^2(s - j\omega)^2 + \omega^2(s + j\omega)^2 + \omega^4}$$

$$= \frac{j}{2}\frac{-j4\omega^2 s}{s^4 + 4\omega^2 s^2} = \frac{2\omega^2}{s(s^2 + 4\omega^2)}$$