EE 362: Signals and Systems

Fourier Analysis of Systems – Chapter 5

Lecture 14

Chapter Contents

- Fourier analysis of continuous-time systems
- Response to periodic and non-periodic inputs
- Analysis of ideal filters
- Sampling theorem
- Fourier analysis of discrete-time systems
- Application to low-pass digital filtering

Fourier analysis of discrete-time systems

Similar to the continuous-time systems,

$$Y(\Omega) = H(\Omega)X(\Omega)$$

$$|Y(\Omega)| = |H(\Omega)||X(\Omega)|$$

$$\angle Y(\Omega) = \angle H(\Omega) + \angle X(\Omega)$$

Response to a Sinusoidal Input

$$x[n] = A\cos(\Omega_0 n + \theta), \quad n = 0, \pm 1, \pm 2, \ldots$$

From table 4.1,

$$X(\Omega) = \sum_{k=-\infty}^{\infty} A\pi[e^{-j\theta}\delta(\Omega + \Omega_0 - 2\pi k) + e^{j\theta}\delta(\Omega - \Omega_0 - 2\pi k)]$$

$$Y(\Omega) = \sum_{k=-\infty}^{\infty} A\pi H(\Omega)[e^{-j\theta}\delta(\Omega + \Omega_0 - 2\pi k) + e^{j\theta}\delta(\Omega - \Omega_0 - 2\pi k)]$$

Now, $H(\Omega)\delta(\Omega + c) = H(-c)\delta(\Omega + c)$ for any constant c, and thus

$$Y(\Omega) = \sum_{k=-\infty}^{\infty} A\pi[H(-\Omega_0 + 2\pi k)e^{-j\theta}\delta(\Omega + \Omega_0 - 2\pi k) + H(\Omega_0 + 2\pi k)e^{j\theta}\delta(\Omega - \Omega_0 - 2\pi k)]$$

Since,

$$H(-\Omega_0) = |H(\Omega_0)|e^{-j\angle H(\Omega_0)} \text{ and } H(\Omega_0) = |H(\Omega_0)|e^{j\angle H(\Omega_0)}$$

Then,

$$Y(\Omega) = \sum_{k=-\infty}^{\infty} A\pi|H(\Omega_0)|[e^{-j(\angle H(\Omega_0)+\theta)}\delta(\Omega + \Omega_0 - 2\pi k) + e^{j(\angle H(\Omega_0)+\theta)}\delta(\Omega - \Omega_0 - 2\pi k)]$$

Taking the inverse DTFT

$$y[n] = A|H(\Omega_0)|\cos(\Omega_0 n + \theta + \angle H(\Omega_0)), \quad n = 0, \pm 1, \pm 2, \ldots$$

Example:

Suppose that $H(\Omega) = 1 + e^{-j\Omega}$ and that the objective is to find the output $y[n]$ resulting from the sinusoidal input $x[n] = 2 + 2\sin\left(\frac{\pi}{2}n\right)$. By linearity, $y[n]$ is equal to the sum of the responses to $x_1[n] = 2$ and $x_2[n] = 2\sin\left(\frac{\pi}{2}n\right)$. The response to $x_1[n] = 2 = 2\cos(0n)$ is

$$y_1[n] = 2|H(0)|\cos(0n + \angle H(0)) = 4$$

The response to $x_2[n] = 2\sin\left(\frac{\pi}{2}n\right)$ is

$$y_2[n] = 2\left|H\left(\frac{\pi}{2}\right)\right| \sin\left(\frac{\pi}{2}n + \angle H\left(\frac{\pi}{2}\right)\right)$$

where $H\left(\frac{\pi}{2}\right) = 1 + e^{-j\pi/2} = \sqrt{2}e^{-j\pi/4}$. Hence, we see that

$$y_2[n] = 2\sqrt{2}\sin\left(\frac{\pi}{2}n - \frac{\pi}{4}\right)$$

Combining $y_1[n]$ and $y_2[n]$ yields

$$y[n] = 4 + 2\sqrt{2}\sin\left(\frac{\pi}{2}n - \frac{\pi}{4}\right)$$

Example MA filter

Consider the N-point MA filter given by the input/output relationship

$$y[n] = \frac{1}{N}\Big[x[n] + x[n-1] + x[n-2] + \cdots + x[n-N+1]\Big]$$

$$Y(\Omega) = \frac{1}{N}[X(\Omega) + X(\Omega)e^{-j\Omega} + X(\Omega)e^{-j2\Omega} \cdots + X(\Omega)e^{-j(N-1)\Omega}]$$

$$Y(\Omega) = \frac{1}{N}[1 + e^{-j\Omega} + e^{-j2\Omega} \ldots + e^{-j(N-1)\Omega}]X(\Omega)$$

From $Y(\Omega) = H(\Omega)X(\Omega)$ and

$$Y(\Omega) = \frac{1}{N}[1 + e^{-j\Omega} + e^{-j2\Omega} \ldots + e^{-j(N-1)\Omega}]X(\Omega)$$

Then,

$$H(\Omega) = \frac{1}{N}[1 + e^{-j\Omega} + e^{-j2\Omega} \ldots + e^{-j(N-1)\Omega}] = \frac{1}{N}\sum_{q=0}^{N-1} e^{-jq\Omega}$$

$$H(\Omega) = \frac{1}{N}\left[\frac{1 - e^{-jN\Omega}}{1 - e^{-j\Omega}}\right]$$

$$H(\Omega) = \frac{1}{N}\left[\frac{e^{-jN\Omega/2}(e^{jN\Omega/2} - e^{-jN\Omega/2})}{e^{-j\Omega/2}(e^{j\Omega/2} - e^{-j\Omega/2})}\right]$$

$$H(\Omega) = \left[\frac{\sin(N\Omega/2)}{N\sin(\Omega/2)}\right]e^{-j(N-1)\Omega/2}$$

$$|H(\Omega)| = \left| \frac{\sin(N\Omega/2)}{N \sin(\Omega/2)} \right|$$

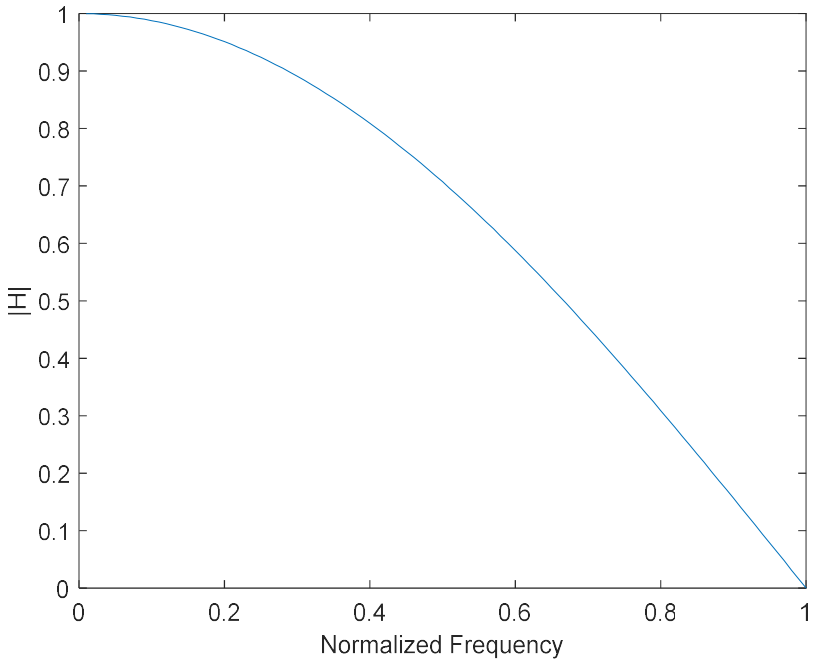
$$\angle H(\Omega) = -\frac{N-1}{2}\Omega, 0 \leq \Omega < \frac{2\pi}{N}$$

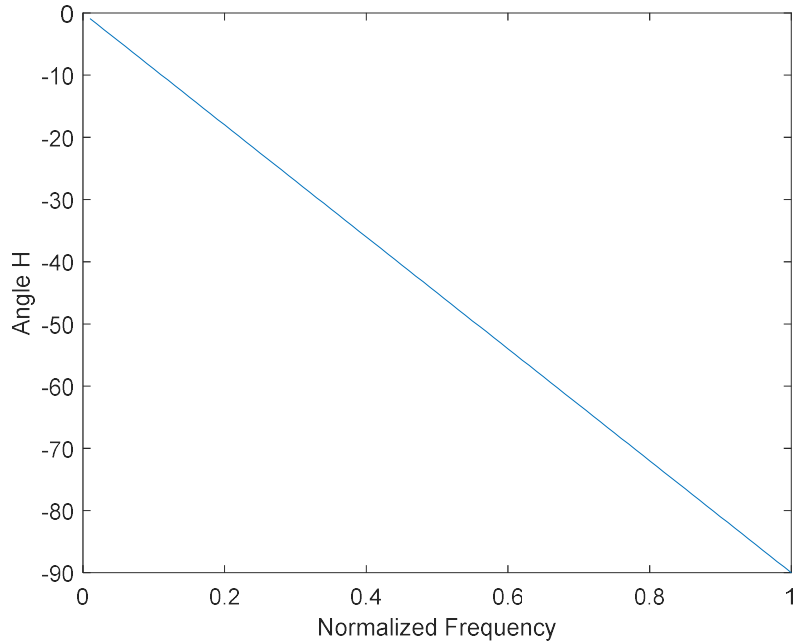
$$\angle H(\Omega) = -\frac{N-1}{2}\Omega + \pi, \frac{2\pi}{N} < \Omega < \frac{4\pi}{N}$$

$$\angle H(\Omega) = -\frac{N-1}{2}\Omega, \frac{4\pi}{N} < \Omega < \frac{6\pi}{N}$$

For any positive integer value of N, the magnitude and phase functions of the MA filter can be computed with MATLAB. For example, when N=2, the MATLAB commands are:

```
w=0:0.01:1;
N=2;
H=1/N.*(1-exp(-j*N*pi*w))./(1-
exp(-j*pi*w));
magH=abs(H);
angH=angle(H)*180/pi;
plot(w,magH)
xlabel('Normalized Frequency')
ylabel('|H|')
figure
plot(w,angH)
xlabel('Normalized Frequency')
ylabel('Angle H')
```


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
|H|
0
0.2
0.4
0.6
0.8
1
Normalized Frequency


0
-10
-20
-30
-40
-50
-60
-70
-80
-90
Angle H
0
0.2
0.4
0.6
0.8
1
Normalized Frequency

In these plots the frequency is normalized frequency (Ω/π radians per unit time). It can be seen that $H(\pi) = 0$.

It is a lowpass discrete-time filter with a linear phase.

Application to Lowpass Digital Filtering

- In chapter 1 it was noted that a signal $x[n]$ with noise can often be expressed in the additive form $x[n] = s[n] + e[n]$, where $s[n]$ is the smooth part of $x[n]$ and $e[n]$ is the erratic or noisy part of $x[n]$.

- The spectrum $E(\Omega)$ of $e[n]$ consists primarily of high-frequency components whose frequencies are in a neighborhood of $\Omega = \pi$.

- The spectrum $S(\Omega)$ of $s[n]$ consists primarily of low-frequency components whose frequencies are in a neighborhood of $\Omega = 0$.

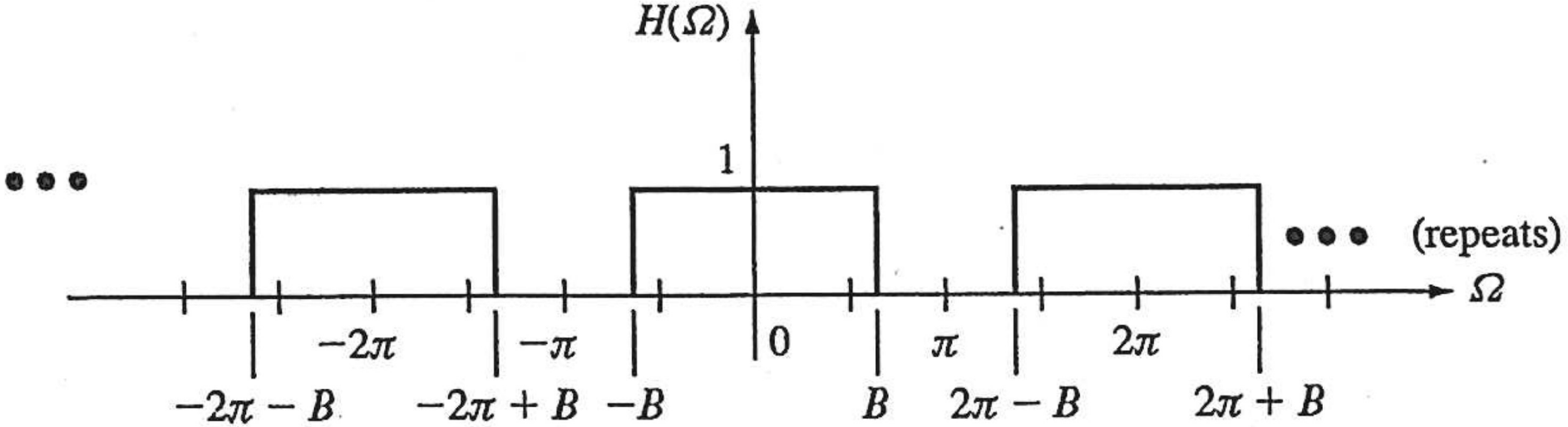
- The objective is to filter $x[n]$ so that the noisy part $e[n]$ is removed, or at least greatly reduced in magnitude.

- This can be done using lowpass filter

- Consider the discrete-time system with the frequency function

$$H(\Omega) = \sum_{k=-\infty}^{\infty} p_{2B}(\Omega + 2\pi k) \quad \text{where } B < \pi.$$

- Note that in this example, the magnitude function $|H(\Omega)|$ is equal to $H(\Omega)$, and the phase function $\angle H(\Omega)$ is identically zero.



- Since $H(\Omega) = 1$ for $0 \leq \Omega_0 \leq B$, the output response for $x[n] = A\cos(\Omega_0 n)\,, n = 0, \pm 1, \pm 2, \ldots,$ is $y[n] = A\cos(\Omega_0 n)\,, n = 0, \pm 1, \pm 2, \ldots,$ for $0 \leq \Omega_0 \leq B$.
- Or, $y[n] = x[n]$ for $0 \leq \Omega_0 \leq B$ and $y[n] = 0$ for $B < \Omega_0 \leq \pi$.

$$y[n] = \begin{cases} A\cos(\Omega_0 n), & 0 \leq \Omega < B \\ 0, & B < \Omega_0 < \pi \end{cases}$$

- This is an ideal lowpass digital filter. Here the term "ideal" refers to the sharp cutoff in $H(\Omega)$ at $\Omega = B$.

- As a result of the periodicity of $H(\Omega)$, the output response $y[n]$ is equal to $A\cos(\Omega_0 n)$ when

$$2\pi k - B \leq \Omega_0 \leq 2\pi k + B, k = 0, 1, 2, \ldots$$

- For all other positive values of Ω_0, the response $y[n]$ is zero.

- As a result of periodicity of the frequency response function $H(\Omega)$, this digital filter is not a "true" lowpass filter, since it passes input sinusoids $A\cos(\Omega_0 n)$ with Ω_0 belonging to the intervals

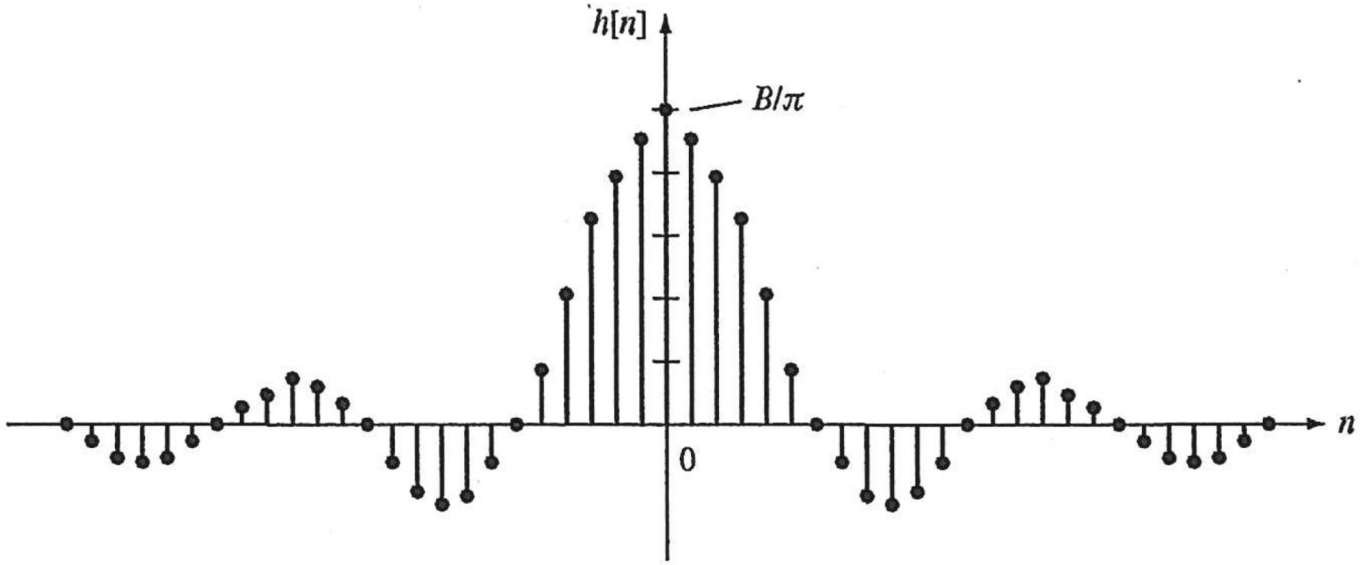
$$2\pi k - B \leq \Omega_0 \leq 2\pi k + B, k = 0, 1, 2, \ldots$$

- If Ω_0 is restricted to lie in the range $0 < \Omega_0 \leq \pi$, the filter can be viewed as an ideal lowpass digital filter with bandwidth B.

Unit-Pulse Response of Ideal Digital Lowpass Filter

- For table 4.1, the unit-pulse response $h[n]$ of the filter with frequency function $H(\Omega) = \sum_{k=-\infty}^{\infty} p_{2B}(\Omega + 2\pi k)$ is

$$h[n] = \frac{B}{\pi}\text{sinc}\left(\frac{B}{\pi}n\right), \quad n = 0, \pm 1, \pm 2, \ldots$$



- It can be seen that $h[n]$ is not zero for $n < 0$, and thus the filter is noncausal.

- Therefore, the filter cannot be implemented online (in real time); but it can be implemented off-line.

- In an off-line implementation, the filtering process is applied to the values of signals that have been stored in the memory of a digital computer or stored by some other means.

- In the discrete-time case, ideal filters can be used in practice, as long as the filtering process is carried out off-line.

- For “real-time filtering” it is necessary to consider a causal lowpass digital filter.

- One very simple example is the 2-point MA filter

$$y[n] = \frac{1}{2}(x[n] + x[n-1])$$

- Weighted Moving Average Filters

$$y[n] = cx[n] + dx[n-1] + fx[n-2]$$

$$c + d + f = 1$$

$$H(\Omega) = c + de^{-j\Omega} + fe^{-j2\Omega}$$