EE 362: Signals and Systems

Fourier Analysis of Systems – Chapter 5

Lecture 13

Chapter Contents

- Fourier analysis of continuous-time systems
- Response to periodic and non-periodic inputs
- Analysis of ideal filters
- Sampling theorem
- Fourier analysis of discrete-time systems
- Application to low-pass digital filtering

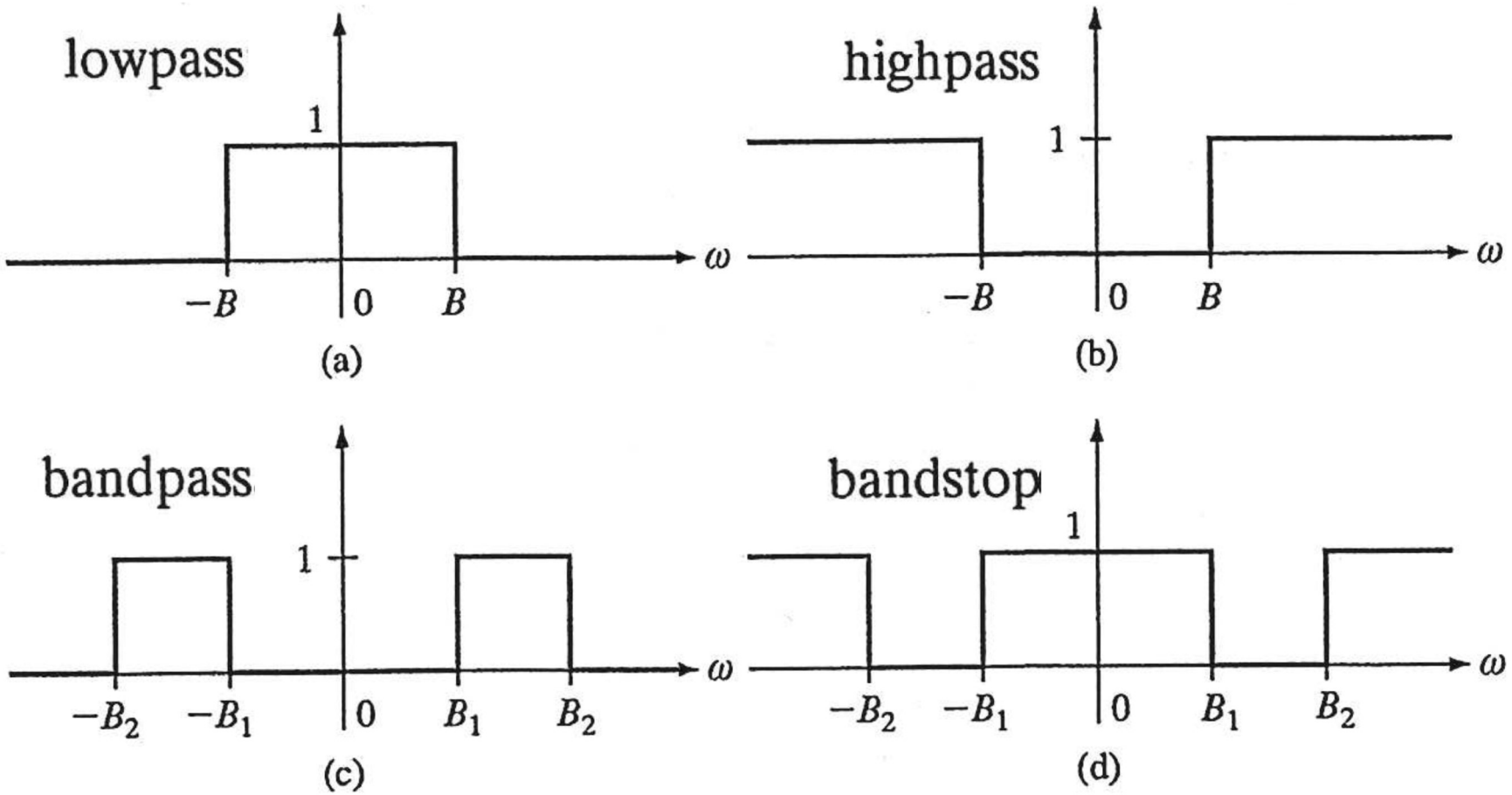
Ideal Filters

Given a linear time-invariant continuous-time system with frequency function $H(\omega)$, it was shown that the output response $y(t)$ resulting from the sinusoidal input $x(t) = A\cos(\omega_0 t + \theta)$ for $-\infty < t < \infty$, is given by

$$y(t) = A|H(\omega_0)|\cos\left(\omega_0 t + \theta + \angle H(\omega_0)\right), -\infty < t < \infty$$

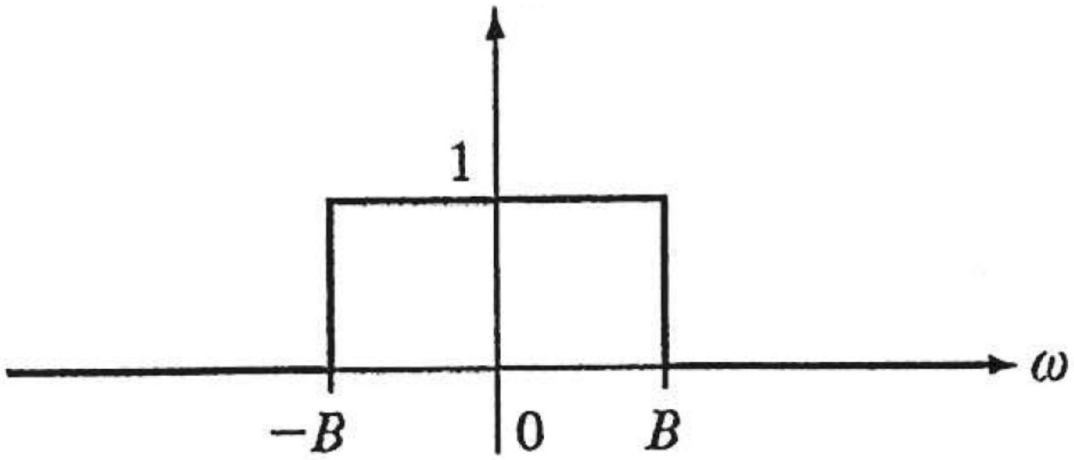
Then, a sinusoid with a particular frequency ω_0 can be prevented from "going through" the system by selecting $H(\omega_0)$ so that $|H(\omega_0)|$ is zero or very small. The process of "rejecting" sinusoids having particular frequencies or range of frequencies is called ***filtering***, and system that has this characteristic is caller ***filter***.

Ideal Filters



The *stopband* of an ideal filter is defined to be the set of all frequencies ω_0 for which the filter completely stops the sinusoidal input $x(t) = A\cos\omega_0 t$, $-\infty < t < \infty$. The *passband* of the filter is the set of all frequencies ω_0 for which the input $x(t)$ is passed without attenuation.

Ideal Low-Pass Filter

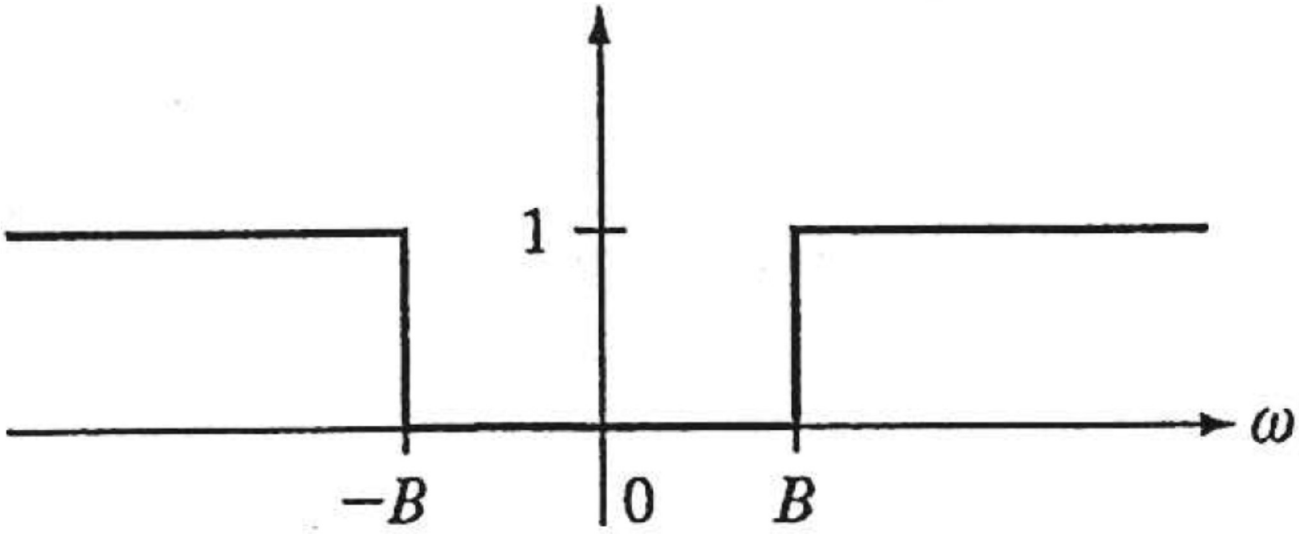


Ideal lowpass: $$|H(\omega)| = \begin{cases} 1, & -B \le \omega \le B \\ 0, & |\omega| > B \end{cases}$$

The frequency range $\omega = 0$ to $\omega = B$ is the passband of the filter, and the range $\omega = B$ to $\omega = \infty$ is the stopband of the filter.

The width B of the passband is defined to be the filter bandwidth.

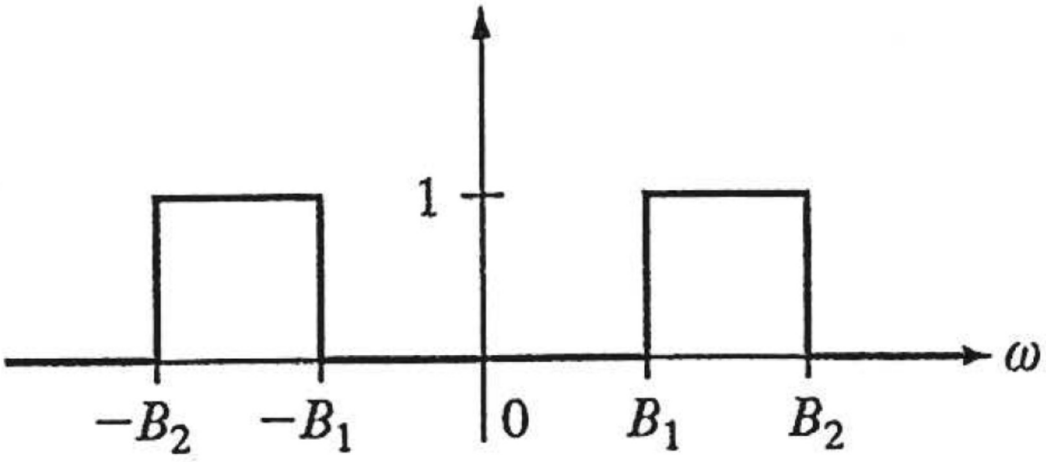
Ideal High-Pass Filter



Ideal highpass: $|H(\omega)| = \begin{cases} 0, & -B < \omega < B \\ 1, & |\omega| \geq B \end{cases}$

The stopband of the highpass filter is the frequency range from $\omega = 0$ to $\omega = B$, and the passband is the frequency range from $\omega = B$ to $\omega = \infty$.

Ideal Band-Pass Filter

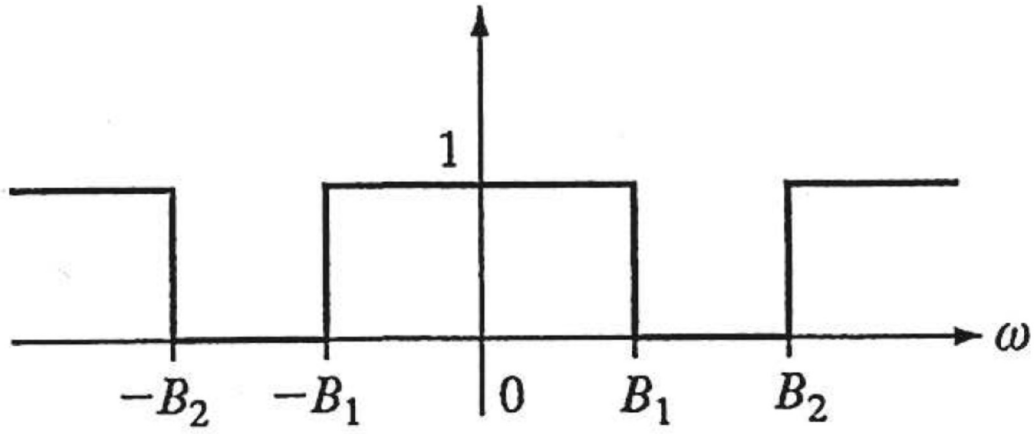


Ideal bandpass: $|H(\omega)| = \begin{cases} 1, & B_1 \le |\omega| \le B_2 \\ 0, & \text{all other } \omega \end{cases}$

the passband of the bandpass filter is the frequency range from $\omega = B_1$ to $\omega = B_2$, while the stopband is the range from $\omega = 0$ to $\omega = B_1$ and the range from $\omega = B_2$ to $\omega = \infty$.

The bandwidth of the bandpass filter is the width of the passband (i.e., $B_2 - B_1$).

Ideal Band-Stop Filter

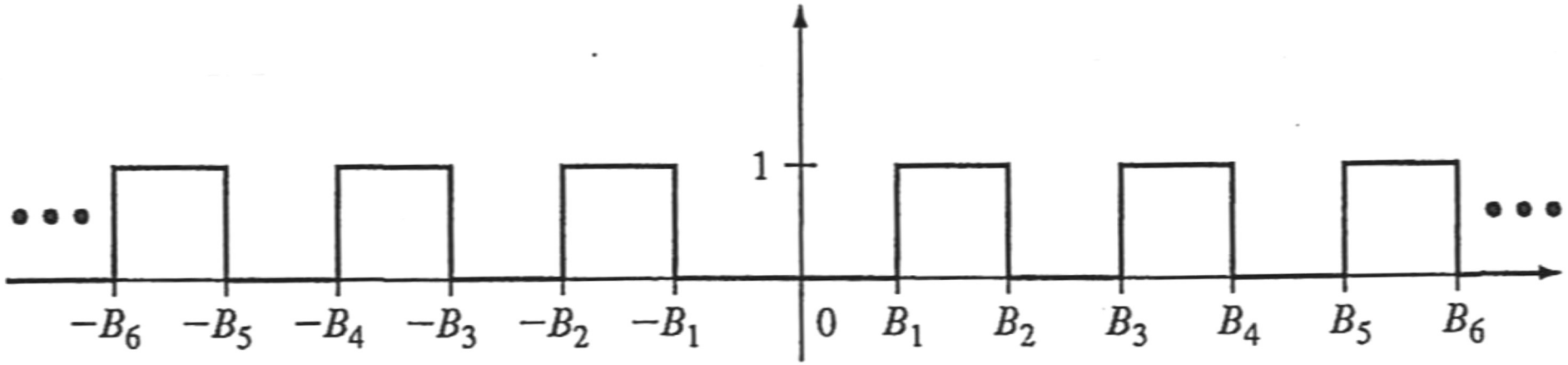


Ideal bandstop: $|H(\omega)| = \begin{cases} 0, & B_1 \le |\omega| \le B_2 \\ 1, & \text{all other } \omega \end{cases}$

the stopband of the bandstop filter is the range from $\omega = B_1$ to $\omega = B_2$, while the passband is the range from $\omega = 0$ to $\omega = B_1$ and the range from $\omega = B_2$ to $\omega = \infty$.

Other Types of Ideal Filters

More complicated examples of ideal filters can be constructed by cascading ideal lowpass, highpass, bandpass, bandstop filters. For instance, by cascading bandstop filters with various values of B_1 and B_2 it is possible to construct an ideal *comb* filter as shown below.



Phase Function

We talked about magnitude of the filters, what about the phase?

To avoid phase distortion in the filtering process, a filter should have a linear phase characteristic over the passband of the filter. In other words, the phase function (in radians) should be of the form

$$\angle H(\omega) = -\omega t_d$$

For all ω in the filter passband where t_d is a fixed positive number.

Now, if ω_0 is in the passband of a linear phase filter, the response resulting from the input $x(t) = A\cos(\omega_0 t)$ for $-\infty < t < \infty$, is given by

$$y(t) = A|H(\omega_0)|\cos(\omega_0 t - \omega t_d), -\infty < t < \infty$$

$$y(t) = A|H(\omega_0)|\cos[\omega_0(t - t_d)], -\infty < t < \infty$$

Then it causes time delay equal to t_d.

Note that if the input is

$$x(t) = A_0 \cos \omega_0 t + A_1 \cos \omega_1 t, -\infty < t < \infty$$

where ω_0 and ω_1 are in the passband of the filter, by linearity the response is

$$y(t) = A_0|H(\omega_0)|\cos[\omega_0(t - t_d)] + A_1|H(\omega_1)|\cos[\omega_1(t - t_d)], -\infty < t < \infty$$

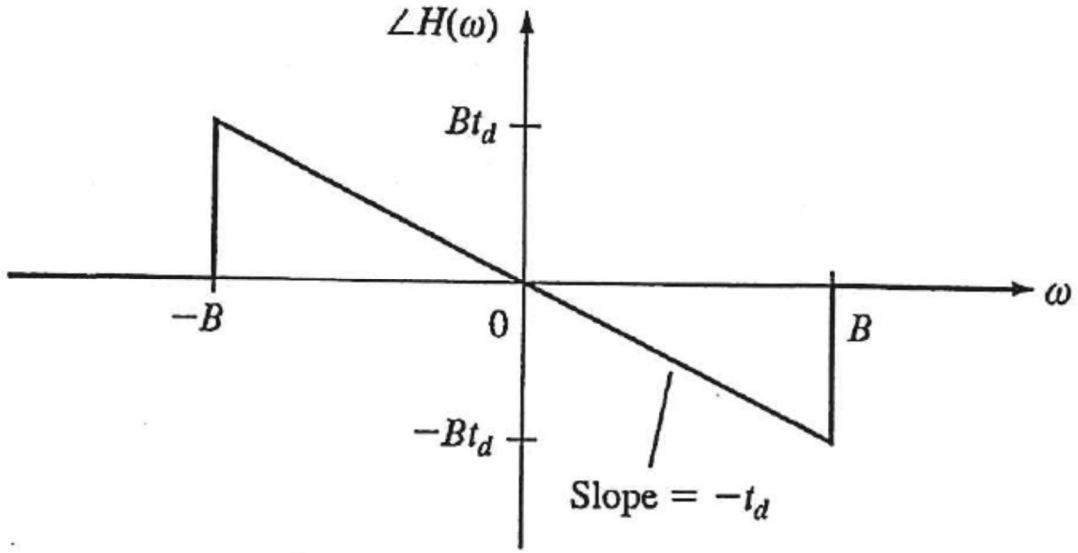
Ideal Linear-Phase Low-Pass Filter

Consider the ideal lowpass filter with the frequency function

$$H(\omega) = \begin{cases} e^{-j\omega t_d}, & -B \le \omega \le B \\ 0, & \omega < -B, \omega > B \end{cases}$$

where t_d is a positive real number.

$$|H(\omega)| = \begin{cases} 1, & -B \le \omega \le B \\ 0, & \omega < -B, \omega > B \end{cases} \quad \angle H(\omega) = \begin{cases} -\omega t_d, & -B \le \omega \le B \\ 0, & \omega < -B, \omega > B \end{cases}$$



Impulse Response of Ideal Linear-Phase Low-Pass Filter

$$H(\omega) = p_{2B}(\omega)e^{-j\omega t_d}, \quad -\infty < \omega < \infty$$

From Table 3.2 the following transform pair can be found:

$$\frac{\tau}{2\pi}\text{sinc}\frac{\tau t}{2\pi} \leftrightarrow p_\tau(\omega)$$

Setting $\tau = 2B$

$$\frac{B}{\pi}\text{sinc}\left(\frac{B}{\pi}t\right) \leftrightarrow p_{2B}(\omega)$$

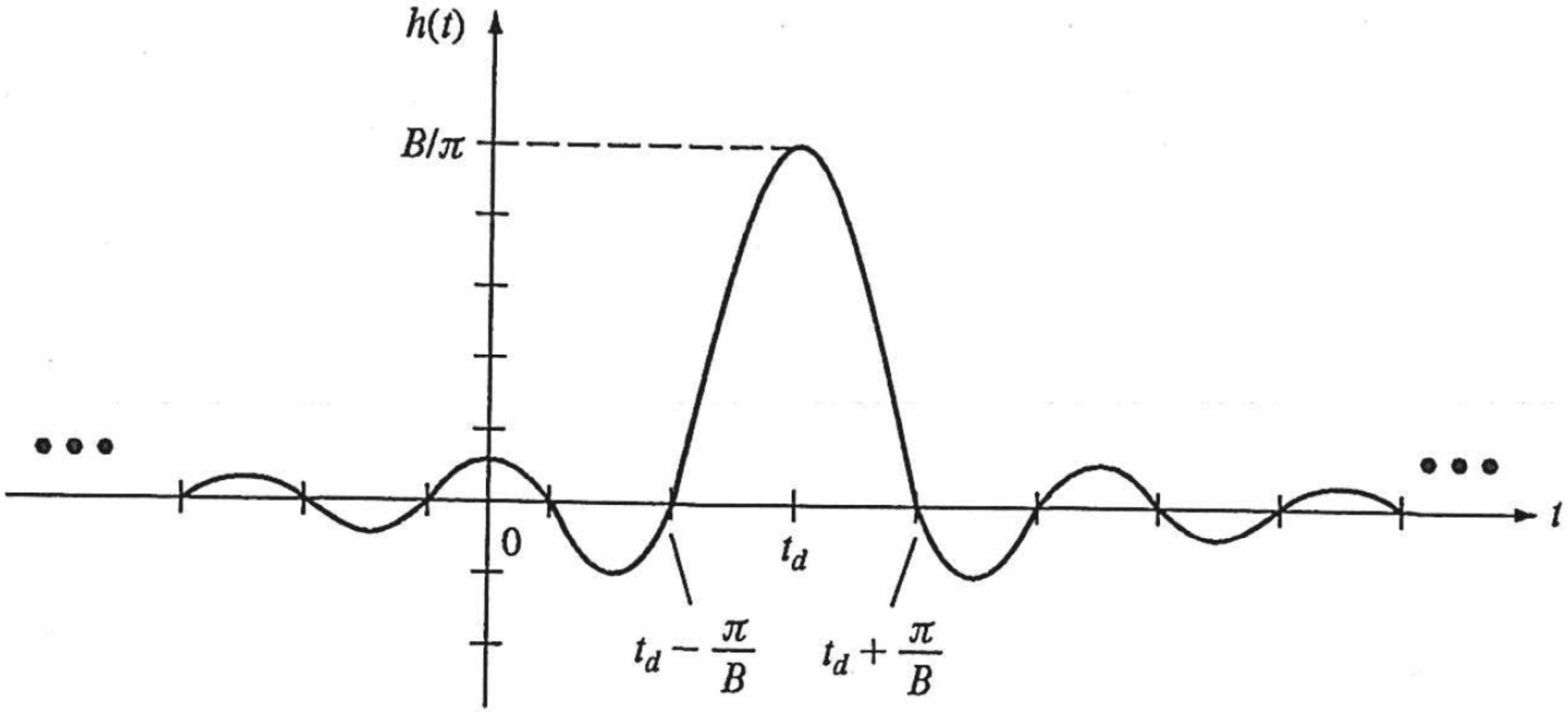
Applying the time-shift property

$$\frac{B}{\pi}\text{sinc}\left[\frac{B}{\pi}(t - t_d)\right] \leftrightarrow p_{2B}(\omega)e^{-j\omega t_d}$$

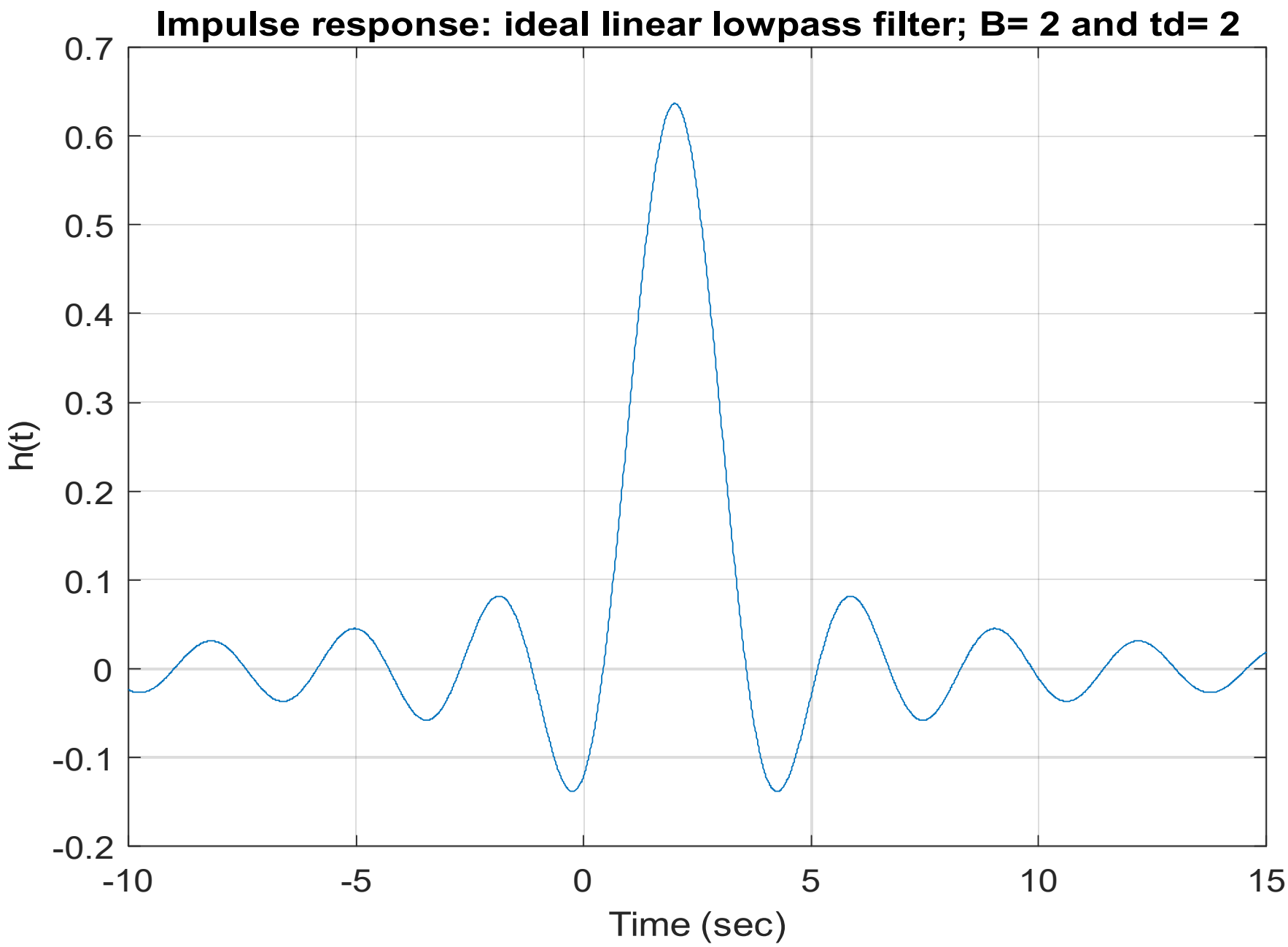
Since the right-hand side of the transform pair is equal to $H(\omega)$,

$$h(t) = \frac{B}{\pi}\text{sinc}\left[\frac{B}{\pi}(t - t_d)\right], \quad -\infty < t < \infty$$

$$h(t) = \frac{B}{\pi}\text{sinc}\left[\frac{B}{\pi}(t - t_d)\right], \quad -\infty < t < \infty$$

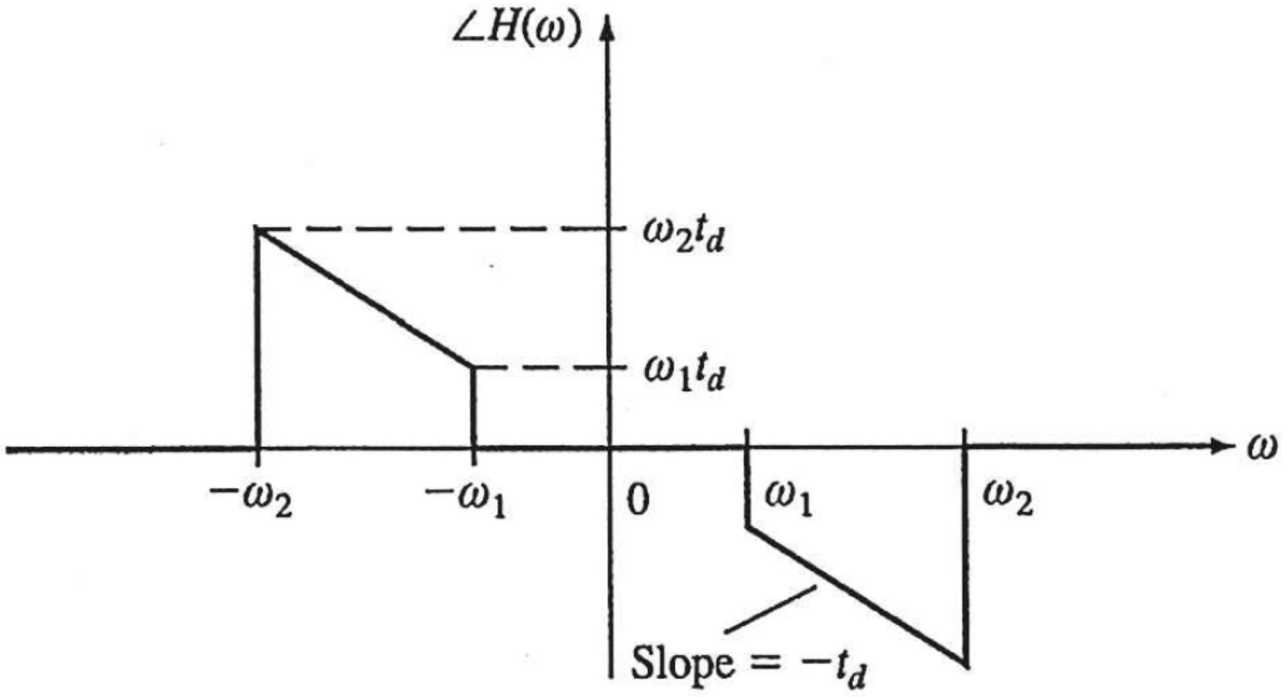


```
t=-10:0.01:15;
B=2;
td=2;
h=B/pi*sinc(B/pi*(t-td));
plot(t,h)
xlabel('Time (sec)')
ylabel('h(t)')
title(sprintf('Impulse response:
ideal linear lowpass filter; B= %s
and td= %s', num2str(B),
num2str(td)))
grid
```


Impulse response: ideal linear lowpass filter; B= 2 and td= 2
h(t)
Time (sec)
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-10
-5
0
5
10
15

Ideal Linear-Phase Band-Pass Filter

$$H(\omega) = \begin{cases} e^{-j\omega t_d}, & B_1 \leq |\omega| \leq B_2 \\ 0, & \text{all other } \omega \end{cases}$$



Sampling Theorem

Sampling is part of a discretization process where a continuous-time signal $x(t)$ is converted into a discrete-time signal $x[n]$.

$$x[n] = x(t)\,|_{t=nT} = x(nT) \qquad n = 0, \pm 1, \pm 2, \ldots$$

To simplify the analysis of the sampling operation, the sampled version of $x(t)$ is often expressed in the form $x(t)p(t)$, where $p(t)$ is the impulse train given by:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Therefore, the sampled signal is:

$$x(t)\,p(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) = \sum_{n=-\infty}^{\infty}x(nT)\delta(t-nT)$$

Note: $\sum_{n=-\infty}^{\infty}\delta(t-nT)$ is a periodic signal. Therefore:

$$p(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT) = \sum_{n=-\infty}^{\infty}c_k e^{jk\omega_s t}$$

Where $\omega_s = 2\pi/T$ is the sampling frequency in rad/sec

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} c_k e^{jk\omega_s t}$$

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t - nT)e^{-jk\omega_s t}\,dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-jk\omega_s t}\,dt = \frac{1}{T}$$

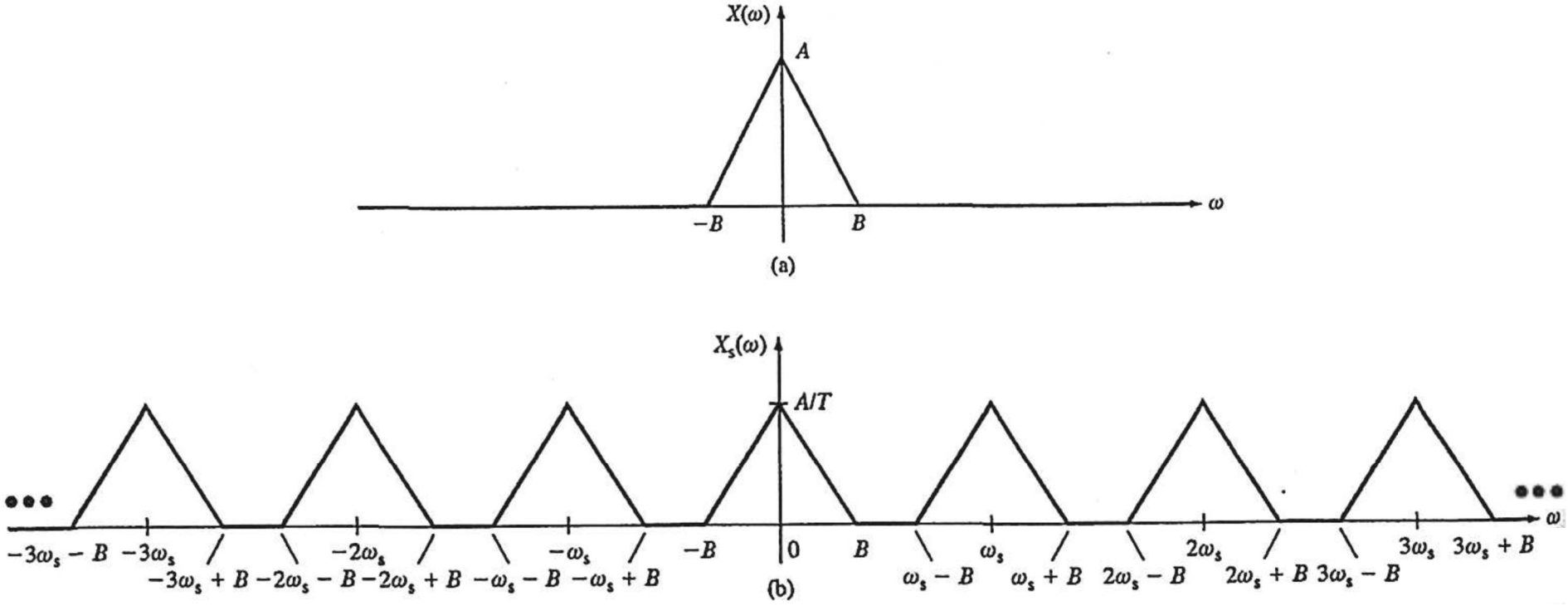
Therefore:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jk\omega_s t} = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jk\omega_s t}$$

$$x(t)p(t) = x(t)\sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T}x(t)\sum_{n=-\infty}^{\infty} e^{jk\omega_s t}$$

Using the property of the Fourier transform involving multiplication complex exponential:

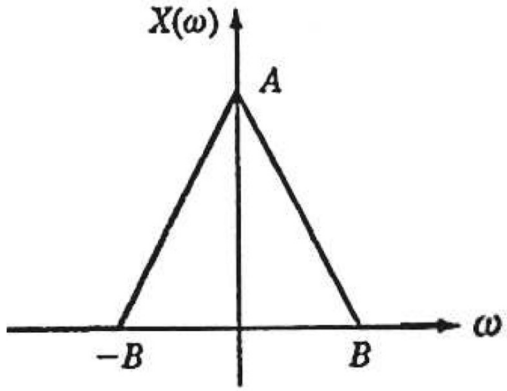
$$x_s(t) = x(t)\,p(t) \overset{f}{\longleftrightarrow} \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) = X_s(\omega)$$



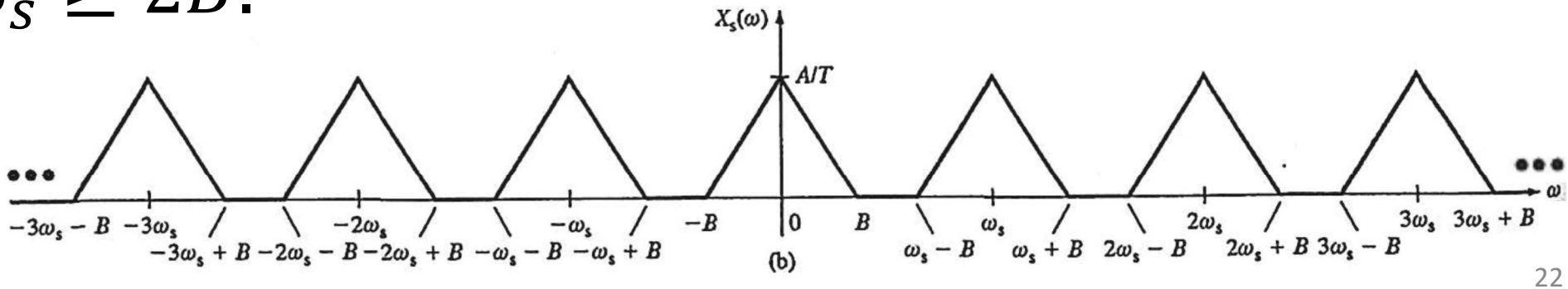
Signal Reconstruction

Given a signal $x(t)$, the reconstruction of $x(t)$ from the sampled waveform $x(t)p(t)$ can be carried out as follows. First, suppose that $x(t)$ has bandwidth B; meaning: $|X(\omega)|=0$ for $\omega>B$

IF $X(\omega)$ has the following form:

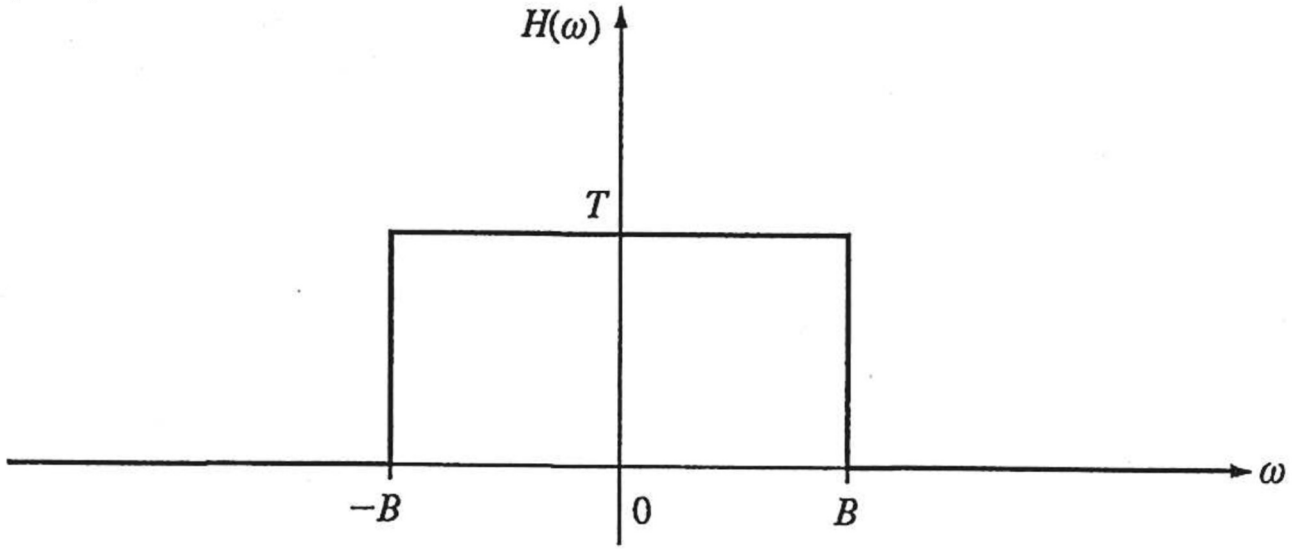


Then, $X_s(\omega)$ has the following form in the case $\omega_s \geq 2B$.



If $\omega_s - B \geq B$ *or* $\omega_s \geq 2B$ for $X_s(\omega)$, the replicas of $X(\omega)$ do not overlap in frequency.

If the sampled signal $x_s(t)$ is applied to an ideal lowpass filter with the frequency function shown below, the only component of $X_s(\omega)$ that is passed is $X(\omega)$. Hence, the output of the filter is equal to $x(t)$.



Then, the original signal $x(t)$ can be completely and exactly reconstructed from the sampled waveform $x_s(t)$. So, the reconstruction of $x(t)$ from the sampled signal $x_s(t) = x(t)p(t)$ can be accomplished by a simple lowpass filtering of the sampled signal. This filter is sometimes called an "interpolation filter," since it produces $x(t)$ from the values of $x(t)$ at the time points $t = nT$.

By this result, which is called the *sampling theorem*, a signal with bandwidth B can be reconstructed completely and exactly from the sampled signal $x_s(t) = x(t)p(t)$ by lowpass filtering with cutoff frequency B if the sampling frequency ω_s is chosen to be greater than or equal to $2B$. The minimum sampling frequency $\omega_s = 2B$ is called the *Nyquist sampling frequency*.

Example: Nyquist Sampling Frequency for Speech

The spectrum of a speech signal is essentially zero for all frequencies above 10 kHz, and so the bandwidth of a speech signal can be taken to be $2\pi \times 10^4$ rad/sec. Then, the Nyquist sampling frequency for speech is

$$\omega_s = 2B = 4\pi \times 10^4 \text{rad/sec}$$

Since $\omega_s = 2\pi/T$, the sampling interval T is equal to $T = 2\pi/\omega_s = 50\ \mu s$. So, the sampling interval corresponding to the Nyquist sampling rate is very small.