EE 362: Signals and Systems

Fourier Analysis of Systems – Chapter 5

Lecture 12

Chapter Contents

- Fourier analysis of continuous-time systems
- Response to periodic and non-periodic inputs
- Analysis of ideal filters
- Sampling theorem
- Fourier analysis of discrete-time systems
- Application to low-pass digital filtering

Fourier analysis of continuous-time systems

- Consider a linear time-invariant (LTI) continuous-time system with impulse response $h(t)$. Then, the output response $y(t)$ resulting from input $x(t)$ is given by the convolution relationship,

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

  - It is not assumed that the system is necessarily causal; then $h(t)$ may be nonzero for $t < 0$.
  - Also, it is assumed that the impulse response $h(t)$ is absolutely integrable, i.e., $\int_{-\infty}^{\infty} |h(t)|dt < \infty$

- The results are not valid in general unless the integrability condition is satisfied.
- As a results of this assumption, the ordinary Fourier transform $H(\omega)$ exists and is given by,

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

Frequency-domain representation

Since $y(t) = h(t) * x(t)$, then

$$\boxed{Y(\omega) = H(\omega)X(\omega)}$$

where $X(\omega)$ is the Fourier transform of the input

- From the ω-domain representation, it is seen that the frequency spectrum $Y(\omega)$ of the output is equal to the product of $H(\omega)$ with the frequency spectrum $X(\omega)$ of the input.

- The magnitude of $Y(\omega)$ is

$$|Y(\omega)| = |H(\omega)||X(\omega)|$$

- The phase of $Y(\omega)$ is

$$\angle Y(\omega) = \angle H(\omega) + \angle X(\omega)$$

Response to a Sinusoidal Input

Suppose the input to the system is

$$x(t) = A\cos(\omega_0 t + \theta) \text{ for } -\infty < t < \infty$$

where A is a positive or negative real number, the frequency ω_0 is assumed to be non-negative, and the phase θ is arbitrary.

From Table 3.2 (in the textbook),

$$X(\omega) = A\pi\left[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)\right]$$

$$Y(\omega) = H(\omega)X(\omega)$$

Then,

$$Y(\omega) = A\pi H(\omega)\left[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)\right]$$

Now, $H(\omega)\delta(\omega + c) = H(-c)\delta(\omega + c)$ for any constant c. Then,

$$Y(\omega) = A\pi\left[H(-\omega_0)e^{-j\theta}\delta(\omega + \omega_0) + H(\omega_0)e^{j\theta}\delta(\omega - \omega_0)\right]$$

Since the impulse response $h(t)$ is real valued, then,

$$|H(-\omega_0)| = |H(\omega_0)|$$

And

$$\angle H(-\omega_0) = -\angle H(\omega_0)$$

Then

$$H(-\omega_0) = |H(\omega_0)|e^{-j\angle H(\omega_0)}$$
$$H(\omega_0) = |H(\omega_0)|e^{j\angle H(\omega_0)}$$

Then,

$$Y(\omega) = A\pi\,|H(\omega_0)|\left[e^{-j(\angle H(\omega_0)+\theta)}\delta(\omega+\omega_0) + e^{j(\angle H(\omega_0)+\theta)}\delta(\omega-\omega_0)\right]$$

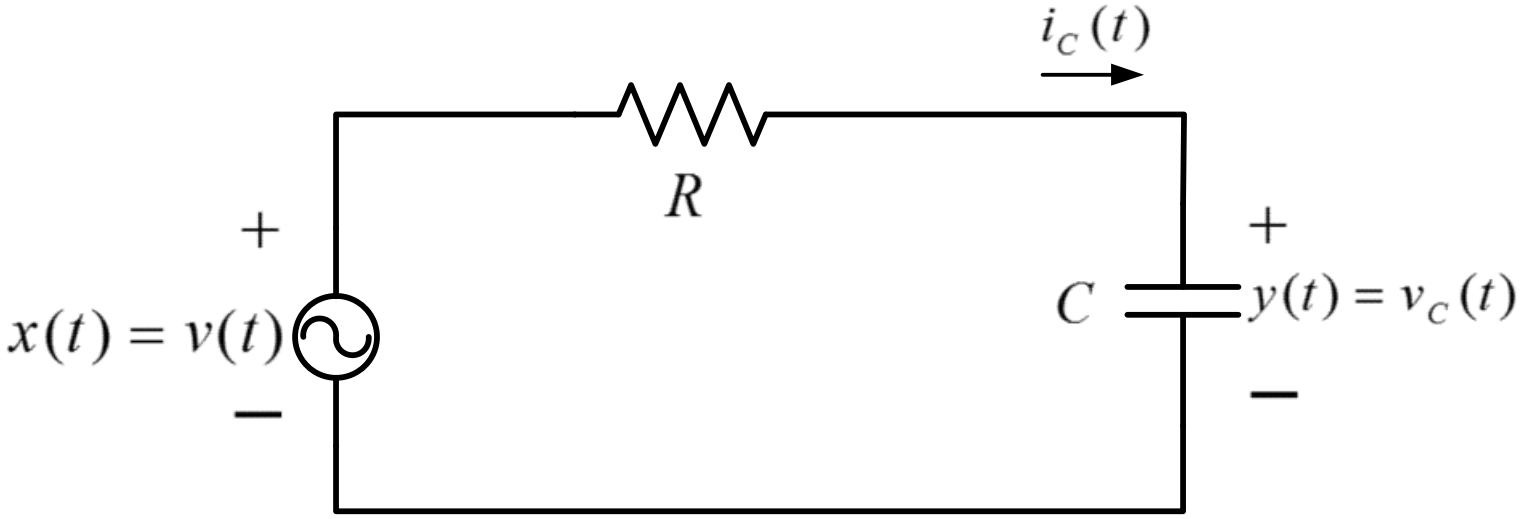
From Table 3.2 (in the textbook),

$$y(t) = A|H(\omega_0)|\cos\left(\omega_0 t + \theta + \angle H(\omega_0)\right), -\infty < t < \infty$$

Then the output is also sinusoid with the same frequency ω_0, but with a magnitude scaled by the factor $|H(\omega_0)|$ and with the phase shifted by amount $\angle H(\omega_0)$.

Example

- Consider the RC circuit shown below,



The input/output differential equation is,

$$-x(t) + Ri_C(t) + y(t) = 0, \qquad \text{but } i_C(t) = C\frac{dy(t)}{dt}$$

$$\therefore \frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

- $\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$

Using derivative property of Fourier transform,

$$j\omega\, Y(\omega) + \frac{1}{RC}Y(\omega) = \frac{1}{RC}X(\omega)$$

$$\therefore Y(\omega) = \frac{1/RC}{j\omega + 1/RC}X(\omega)$$

But,

$$Y(\omega) = H(\omega)X(\omega)$$

Then,

$$H(\omega) = \frac{1/RC}{j\omega + 1/RC}$$

Frequency response function or Transfer Function

The magnitude of $H(\omega)$ is,

$$|H(\omega)| = \frac{1/RC}{\sqrt{\omega^2 + (1/RC)^2}}$$

The phase of $H(\omega)$ is,

$$\angle H(\omega) = -\tan^{-1} \omega RC$$

If we choose $1/RC = 1000$, then

$$|H(0)| = 1, \qquad \angle H(0) = 0$$

$$|H(1000)| = 1/\sqrt{2}, \qquad \angle H(1000) = -45^o$$

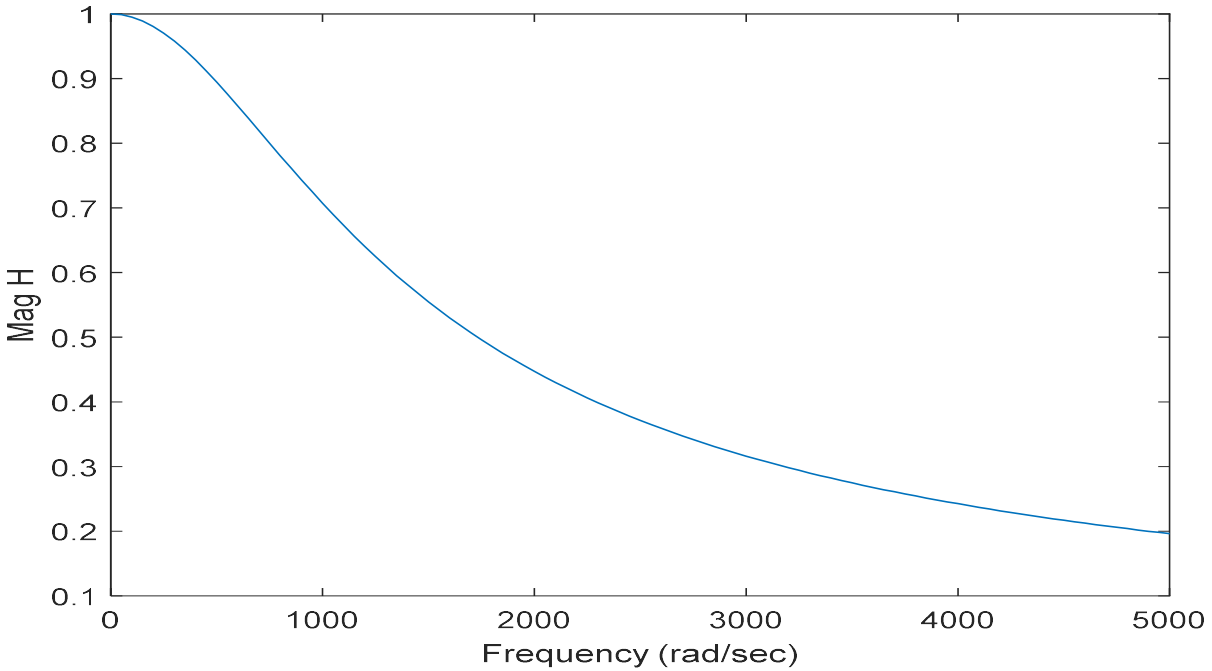
$$|H(3000)| = 0.316, \qquad \angle H(3000) = -71.6^o$$

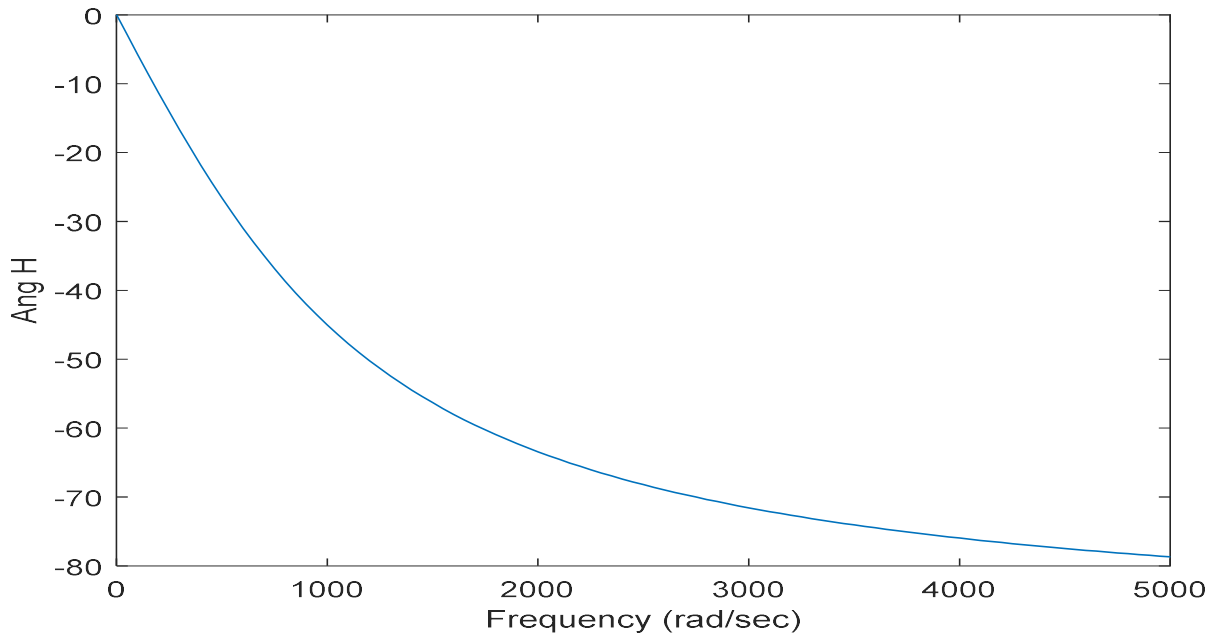
$$|H(\omega)| \to 0 \; as \quad \omega \to \infty$$

$$\angle H(\omega) \to -90^o as \quad \omega \to \infty$$

Plotting using MATLAB,

```
RC=1/1000;
w=0:50:5000;
H=(1/RC)./(j*w+1/RC);
MagH=abs(H);
AngH=angle(H)*180/pi;
plot(w,MagH)
xlabel('Frequency (rad/sec)')
ylabel('Mag H')
figure
plot(w,AngH)
xlabel('Frequency (rad/sec)')
ylabel('Ang H')
```


Mag H
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1000
2000
3000
4000
5000
Frequency (rad/sec)


Ang H
0
-10
-20
-30
-40
-50
-60
-70
-80
0
1000
2000
3000
4000
5000
Frequency (rad/sec)

If the input is $x(t) = A\cos(\omega_0 t + \theta)$ and $1/RC = 1000$, then if $\omega_0 = 0$,

$$y(t) = A\,(1)\cos(0 + \theta + 0) = A\cos(\theta)$$

for $-\infty < t < \infty$

If $\omega_0 = 1000$ (rad/sec),

$$y(t) = A\,(0.707)\cos(1000t + \theta - 45^o)$$

for $-\infty < t < \infty$

If $\omega_0 = 3000$ (rad/sec),

$$y(t) = A\,(0.316)\cos(3000t + \theta - 71.6^o)$$

for $-\infty < t < \infty$

Finally, the output response goes to zero as $\omega \to \infty$

- Then, when $\omega_0 = 0$, the output is equal to the input $x(t) = A\cos(\theta)$ and $y(t) = A\cos(\theta)$
- Hence, the RC circuit passes a dc input without attenuation and without phase shift.
- As ω_0 increases, the magnitude is increasingly attenuated (goes to zero as $\omega \to \infty$) and the phase is increasingly shifted (goes to -90^o as $\omega \to \infty$)
- Then the RC circuit "passes" low-frequency signals without significant attenuation and without producing any significant phase shift.
- As $\omega \to \infty$, the RC circuit completely "blocks" the input.
- **This is called low-pass filter**

- Now suppose that the input is the sum of two sinusoids,

$$x(t) = A_1 \cos(\omega_1 t + \theta_1) + A_2 \cos(\omega_2 t + \theta_2)$$

Then,

$$y(t) = A_1 |H(\omega_1)| \cos\left(\omega_1 t + \theta_1 + \angle H(\omega_1)\right) + A_2 |H(\omega_2)| \cos\left(\omega_2 t + \theta_2 + \angle H(\omega_2)\right)$$

Now consider

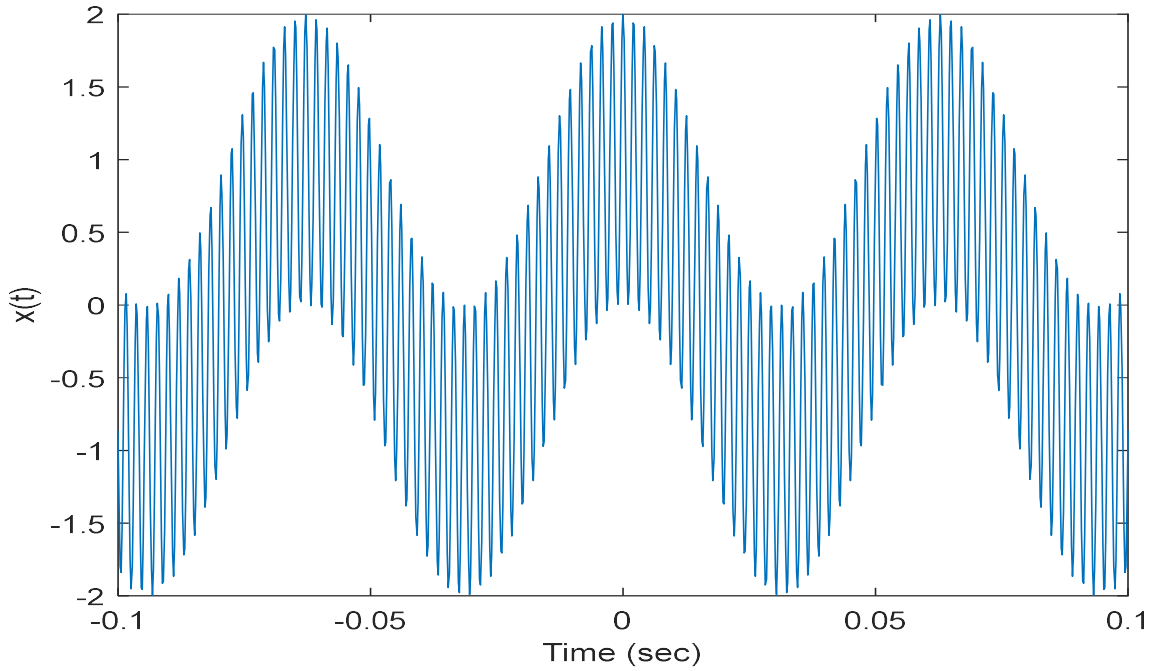
$$x(t) = A_1 \cos(100t) + A_2 \cos(3000t)$$

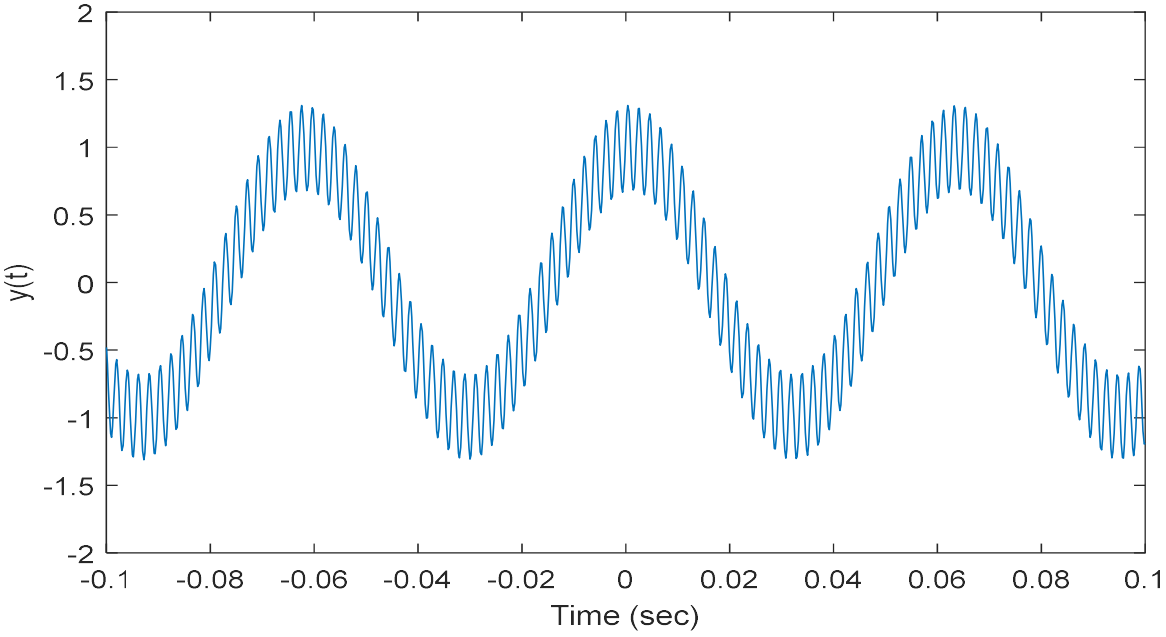
and $1/RC = 1000$

MATLAB plots follow

MATLAB

```
RC=1/1000;
t=-0.1:0.2/1000:0.1;
w1=100; w2=3000;
Hw1=(1/RC)./(j*w1+1/RC);
Hw2=(1/RC)./(j*w2+1/RC);
x=cos(w1*t)+cos(w2*t);
y=abs(Hw1)*cos(w1*t+angle(Hw1))+abs(Hw2)*cos(
w2*t+angle(Hw2));
plot(t,x), xlabel('Time (sec)'),
ylabel('x(t)')
figure
plot(t,y), xlabel('Time (sec)'),
ylabel('y(t)')
```


x(t)
Time (sec)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-0.1
-0.05
0
0.05
0.1


y(t)
Time (sec)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-0.1
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
0.1

Response to periodic and non-periodic inputs

For a periodic signal, the Fourier series can be expressed as follows,

$$x(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k), -\infty < t < \infty$$

where

$$A_k = \sqrt{a_k^2 + b_k^2}, \qquad k = 1, 2, \ldots$$

$$\theta_k = \begin{cases} tan^{-1}\left(\frac{-b_k}{a_k}\right), & k = 1, 2, \ldots, when\ a_k \geq 0 \\ \pi + tan^{-1}\left(\frac{-b_k}{a_k}\right), & k = 1, 2, \ldots, when\ a_k < 0 \end{cases}$$

The output response resulting from the **sinusoidal** input $A_k \cos(k\omega_0 t + \theta_k)$ is

$$y(t) = A_k |H(k\omega_0)| \cos\left(k\omega_0 t + \theta_k + \angle H(k\omega_0)\right)$$

Also, the response due to a constant input a_0 is $a_0 H(0)$. Then, by linearity, the response to the periodic input $x(t)$ for $-\infty < t < \infty$ is,

$$y(t) = a_0 H(0) + \sum_{k=1}^{\infty} A_k |H(k\omega_0)| \cos\left(k\omega_0 t + \theta_k + \angle H(k\omega_0)\right)$$

Then the response is periodic too and having the same period (T) and frequency (ω_0) as the input.

Now, let A_k^x and θ_k^x denote the coefficients of the trigonometric Fourier series for $x(t)$ and A_k^y and θ_k^y denote the coefficients of the trigonometric Fourier series for the resulting output $y(t)$, then

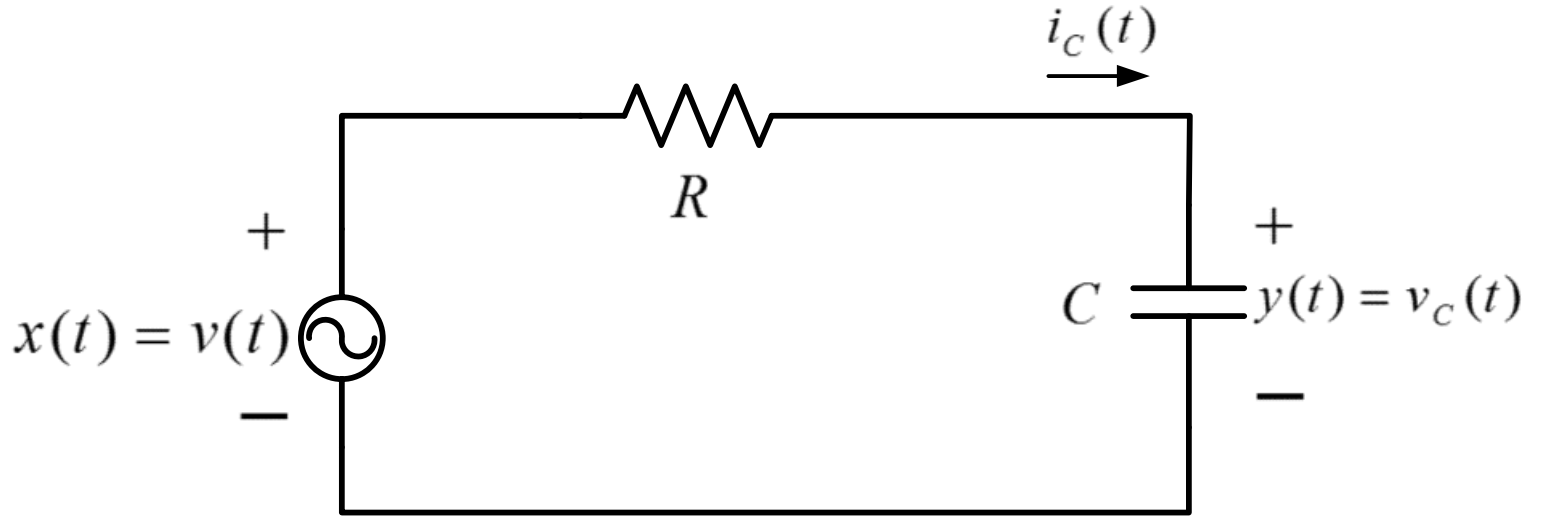
$$A_k^y = A_k^x |H(k\omega_0)|$$
$$\theta_k^y = \theta_k^x + \angle H(k\omega_0)$$

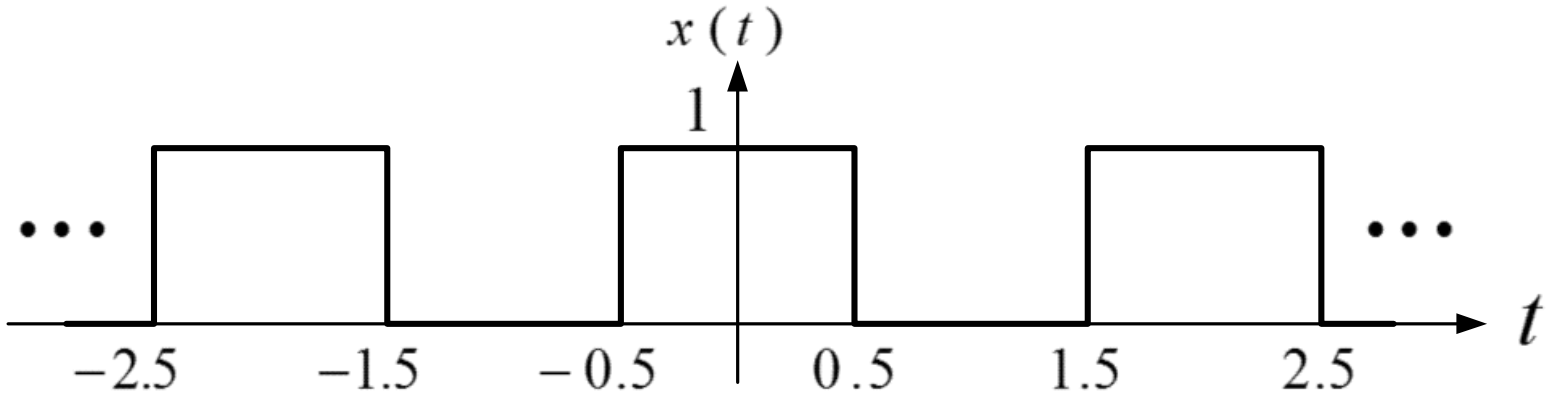
Thus, the Fourier series of the output can be calculated directly from the coefficients of the input Fourier series.

- From Eq. (3.27) and the above expressions, the complex Fourier series can be expresses as follows

$$|c_k^y| = \frac{1}{2}|H(k\omega_0)|A_k^x \quad \text{and} \quad \angle c_k^y = \theta_k^x + \angle H(k\omega_0)$$

Example: Consider the circuit shown below. The objective is to determine the $y(t)$ on the capacitor resulting from the rectangular pulse train $x(t)$.





From chapter 3 we know that the trigonometric Fourier series of this pulse train is,

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\pi t), -\infty < t < \infty$$

where $a_0 = 0.5$ and

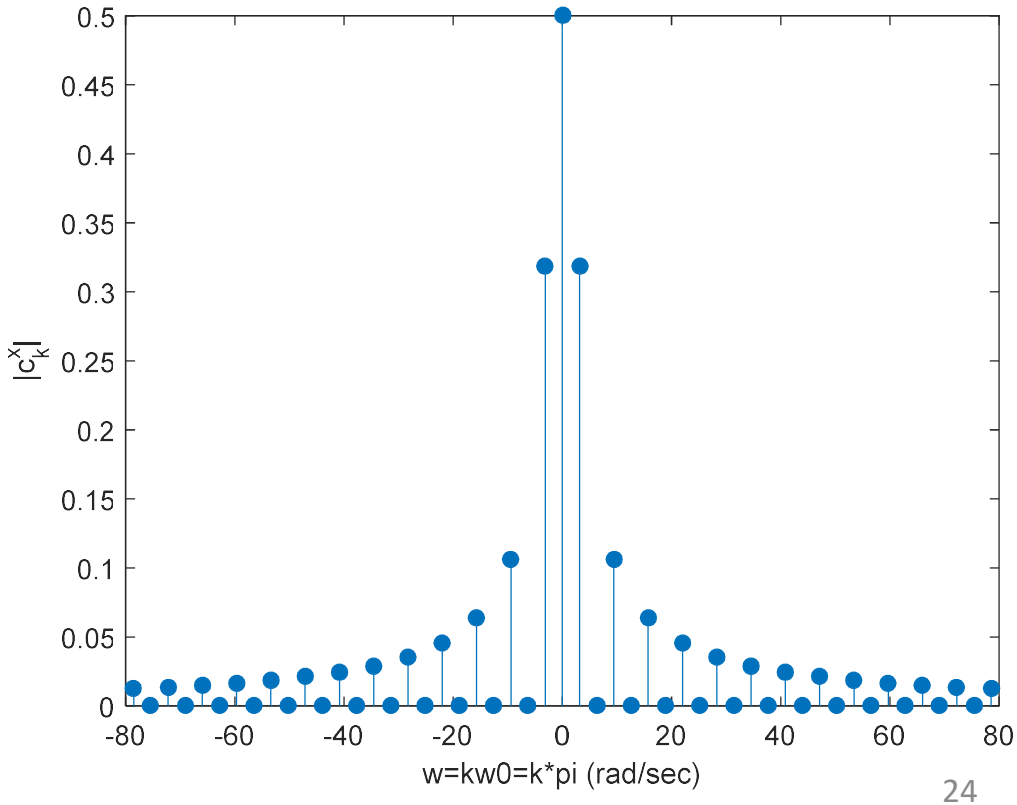
$$a_k = \frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right) = \begin{cases} \frac{2}{k\pi}(-1)^{(k-1)/2}, & k = 1, 3, 5, \ldots \\ 0, & k = 2, 4, 6, \ldots \end{cases}$$

$$|c_k^x| = \begin{cases} 0.5, & k = 0 \\ \frac{1}{k\pi}, & k = \pm 1, \pm 3, \pm 5, \ldots \\ 0, & k = \pm 2, \pm 4, \pm 6, \ldots \end{cases}$$

The amplitude spectrum of $|c_k^x|$ versus $\omega = k\omega_0 = k\pi$.

The plot shows that $x(t)$ has frequency component from dc all the way to infinite frequency, with the higher-frequency components having less significance.

```
k=-25:25;
c=mod(k,2)./(abs(k)*pi);
c(find(k==0))=0.5;
w=k*pi;
stem(w,c,'fill')
xlabel('w=kw0=k*pi
(rad/sec)')
ylabel('|c^x_k|')
```



From the previous lecture,

$$H(\omega) = \frac{1/RC}{j\omega + 1/RC}$$

Hence,

$$H(k\omega_0) = H(k\pi) = \frac{1/RC}{jk\pi + 1/RC}$$

and

$$|H(k\pi)| = \frac{1/RC}{\sqrt{k^2\pi^2 + (1/RC)^2}}$$

$$\angle H(k\pi) = -\tan^{-1} k\pi RC$$

$H(k\pi)$ rolls off as the integer k increases

$1/RC$ is the corner frequency or the 3dB bandwidth

The cosine-with-phase from of the Fourier series,

$$A_k^x = \begin{cases} \dfrac{2}{k\pi}, & k = 1, 3, 5, \ldots \\ 0, & k = 2, 4, 6, \ldots \end{cases}$$

$$\theta_k^x = \begin{cases} \pi, & k = 3, 7, 11, \ldots \\ 0, & \text{all other } k \end{cases}$$

Then, $a_0^y = H(0)a_0^x = 0.5$ and

$$A_k^y = \begin{cases} \dfrac{2}{k\pi} \dfrac{1/RC}{\sqrt{k^2\pi^2 + (1/RC)^2}}, & k\ odd \\ 0, & k\ even \end{cases}$$

$$\theta_k^y = \begin{cases} \pi - \tan^{-1} k\pi RC\,, & k = 3, 7, 11, \ldots \\ -\tan^{-1} k\pi RC\,, & \text{all other } k \end{cases}$$

We can see the effect of the bandwidth of the circuit on the output by plotting the amplitude spectrum ($|c_k^x| = \frac{1}{2}|c_k^x|$ versus $\omega = k\pi$) of the output for various values of $1/RC$.

The next slide shows the cases where $1/RC = 1$, $1/RC = 10$, and $1/RC = 100$.

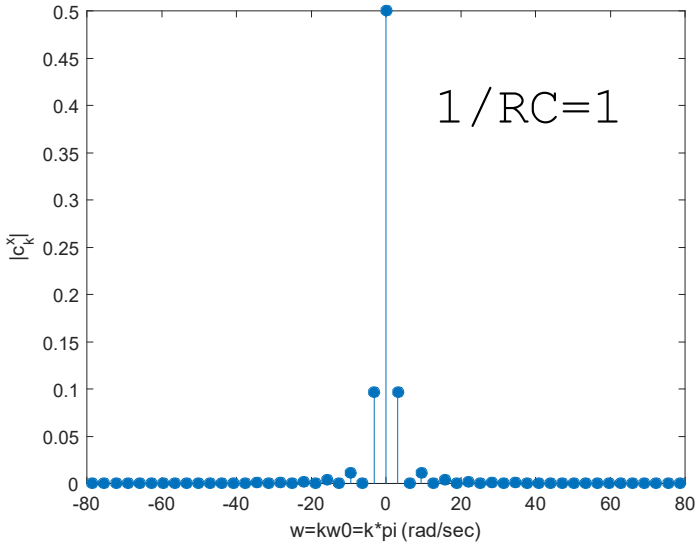
When $1/RC = 1$, the circuit attenuates much of the spectral content.

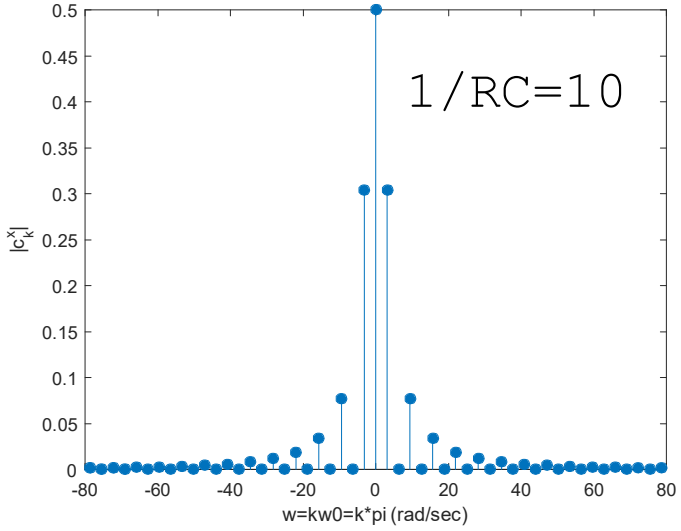
When $1/RC = 100$, there is very little attenuation.

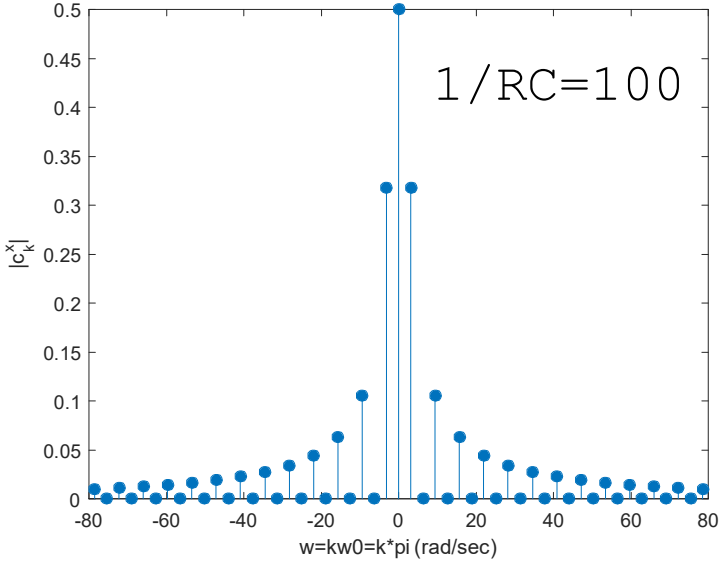
Then, when $1/RC$ is small the circuit significantly distorts the pulse train and whereas when$1/RC$ is large, there should not be much distortion.

Verification next.

```
RC=1; % change it to 1/10 and 1/100 and observe the change
k=-25:25;
c=mod(k,2)*(1/2)*2./(abs(k)*pi).*(1/RC)./sqrt((abs(k)).^2*pi^2
+(1/RC)^2);
c(find(k==0))=0.5;
w=k*pi;
stem(w,c,'fill')
xlabel('w=kw0=k*pi (rad/sec)')
ylabel('|c^x_k|')
```







To verify this, let's plot the output.

$$y(t) = 0.5 + \sum_{\substack{k=1 \\ K\ odd}}^{\infty} \frac{2}{k\pi} \frac{1/RC}{\sqrt{k^2\pi^2 + (1/RC)^2}} \cos\left(k\pi t + \theta_k^y\right)$$

From $\theta_k^y = \begin{cases} \pi - \tan^{-1} k\pi RC\,, & k = 3, 7, 11, \ldots \\ -\tan^{-1} k\pi RC\,, & \text{all other } k \end{cases}$,

$$\cos\left(k\pi t + \theta_k^y\right) = (-1)^{(k-1)/2} \cos(k\pi t - \tan^{-1} k\pi RC)$$

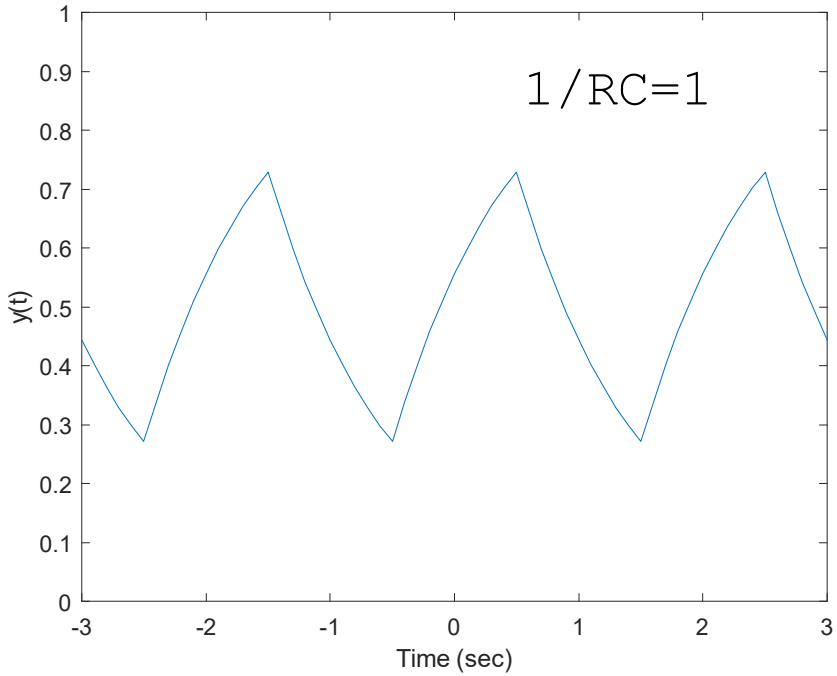
for $k = 1, 3, 5, \ldots$. Then,

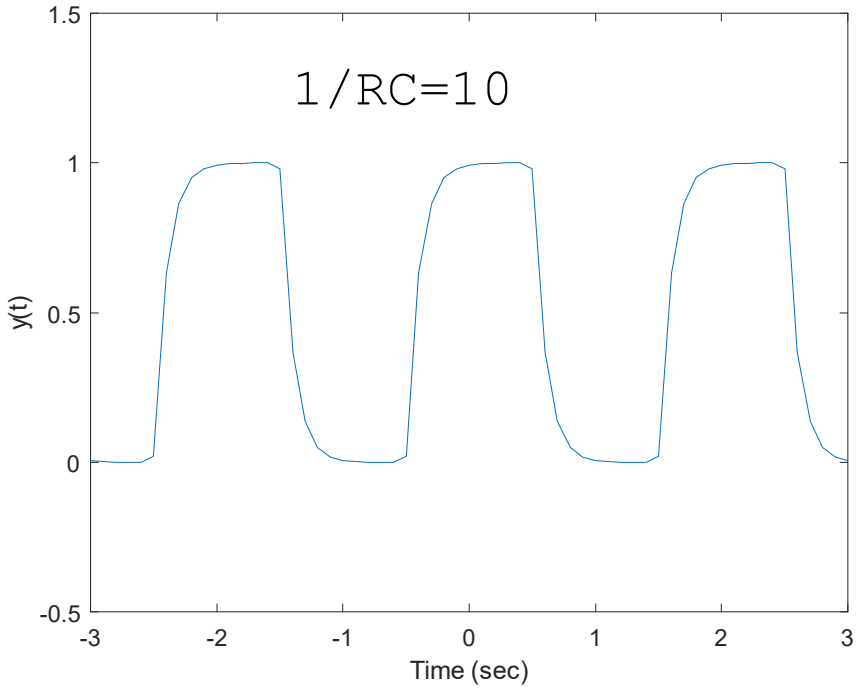
$$y(t) = 0.5 + \sum_{\substack{k=1 \\ K\ odd}}^{\infty} \frac{2}{k\pi} (-1)^{(k-1)/2} \frac{1/RC}{\sqrt{k^2\pi^2 + (1/RC)^2}} \cos(k\pi t - \tan^{-1} k\pi RC)$$

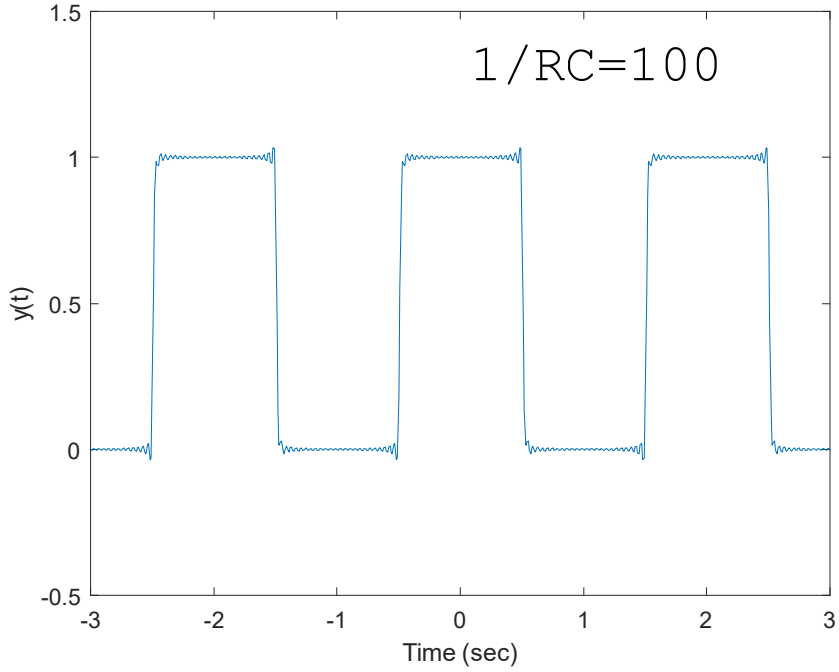
Plotting of $y(t)$

```
RC=1; a0=0.5; H0=1; N=50; w0=pi;
t=-3:0.01:3;
y=a0*H0*ones(1,length(t));
for k=1:2:N
    Bk=2/pi/k*(-1)^((k-1)/2);
    H=(1/RC)/(j*k*w0+1/RC);
    y=y+Bk*abs(H)*cos(k*w0*t+angle(H));
end
plot(t,y)
xlabel('Time (sec)')
ylabel('y(t)')
axis([-3,3,-0.5,1.5])
```

Try `RC=1/10` and `RC=1/100`


1/RC=1
y(t)
Time (sec)


1/RC=10
y(t)
Time (sec)


1/RC=100
y(t)
Time (sec)

Response to Nonperiodic Inputs

We can compute the response of $y(t)$ resulting from input $x(t)$ by:

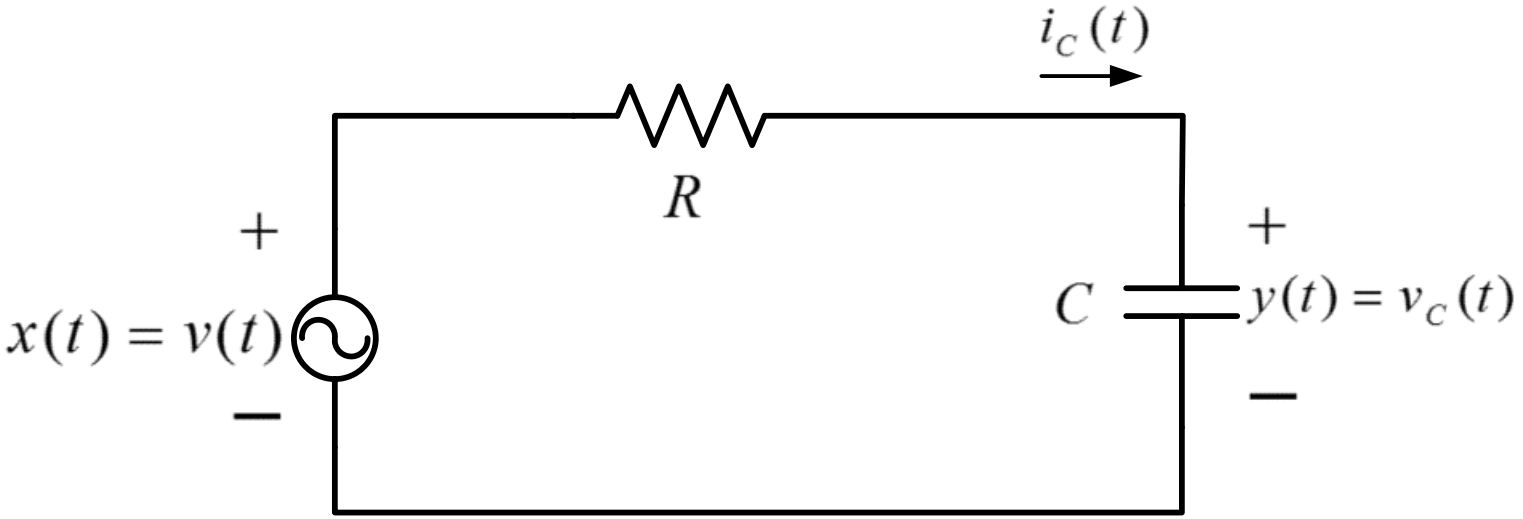
1- Find Fourier transform $Y(\omega)$ of the $y(t)$ using

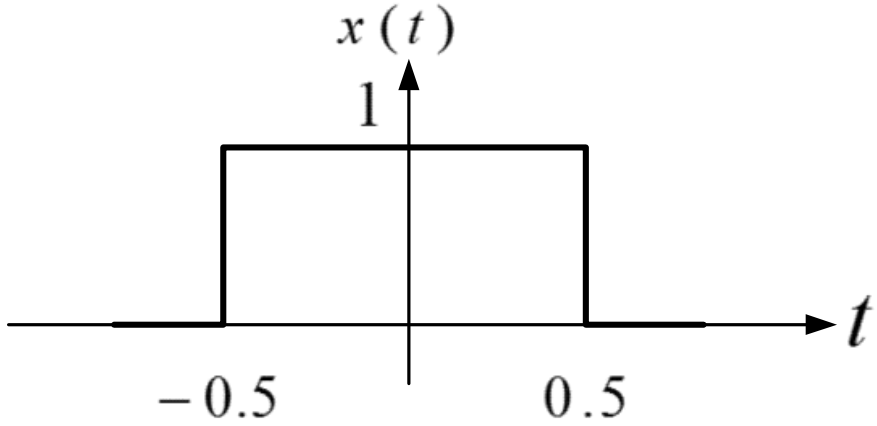
$$Y(\omega) = H(\omega)X(\omega)$$

2- Compute the output by taking the inverse Fourier transform of $H(\omega)X(\omega)$; that is,

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)X(\omega)e^{j\omega t}d\omega$$

Example: Consider the circuit shown below. The objective is to determine the $y(t)$ on the capacitor resulting from the rectangular pulse $x(t)$.

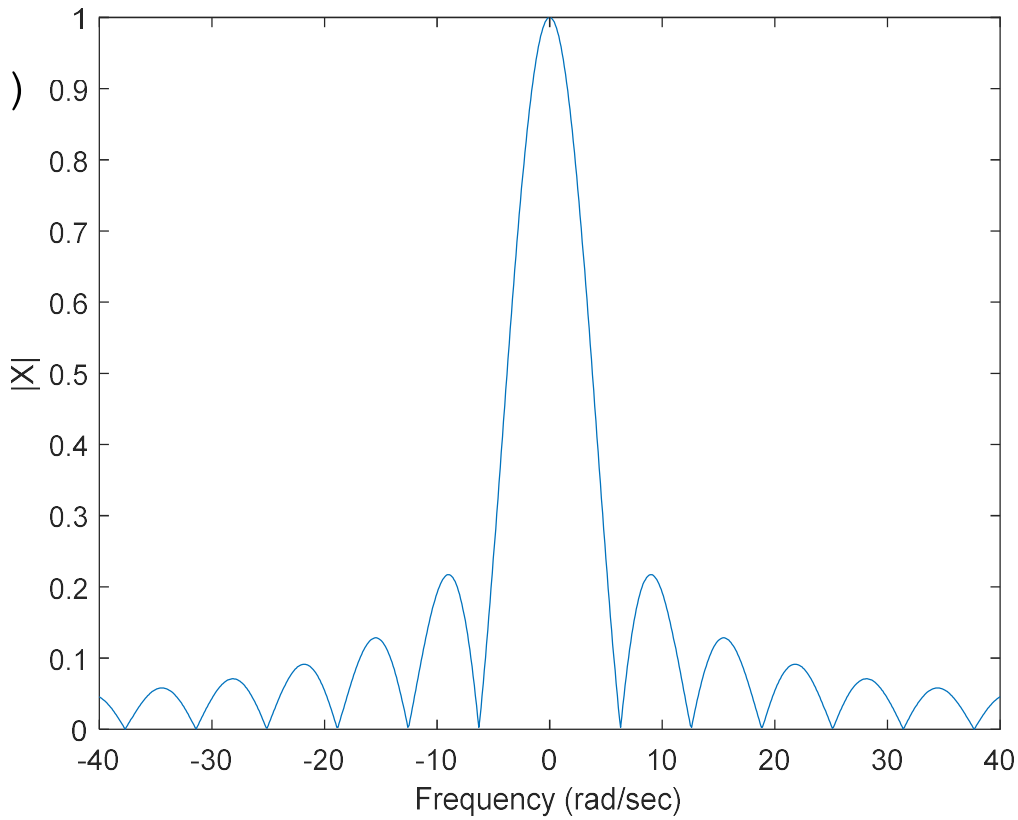




The Fourier transform of x(t) is

$$X(\omega) = sinc\left(\frac{\omega}{2\pi}\right) = \frac{\sin(\omega/2)}{\omega/2} = 2\frac{\sin(\omega/2)}{\omega}$$

```
w=-40:0.1:40;
X=2*sin(w/2)./w;
plot(w,abs(X))
xlabel('Frequency (rad/sec)')
ylabel('|X|')
```



The Fourier transform of the output is

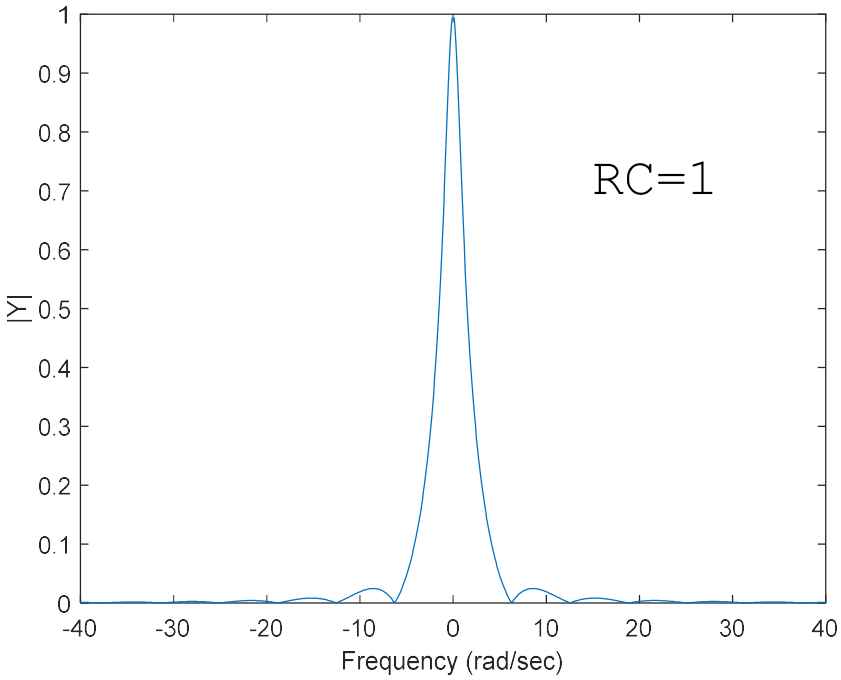
$$Y(\omega) = H(\omega)X(\omega)$$

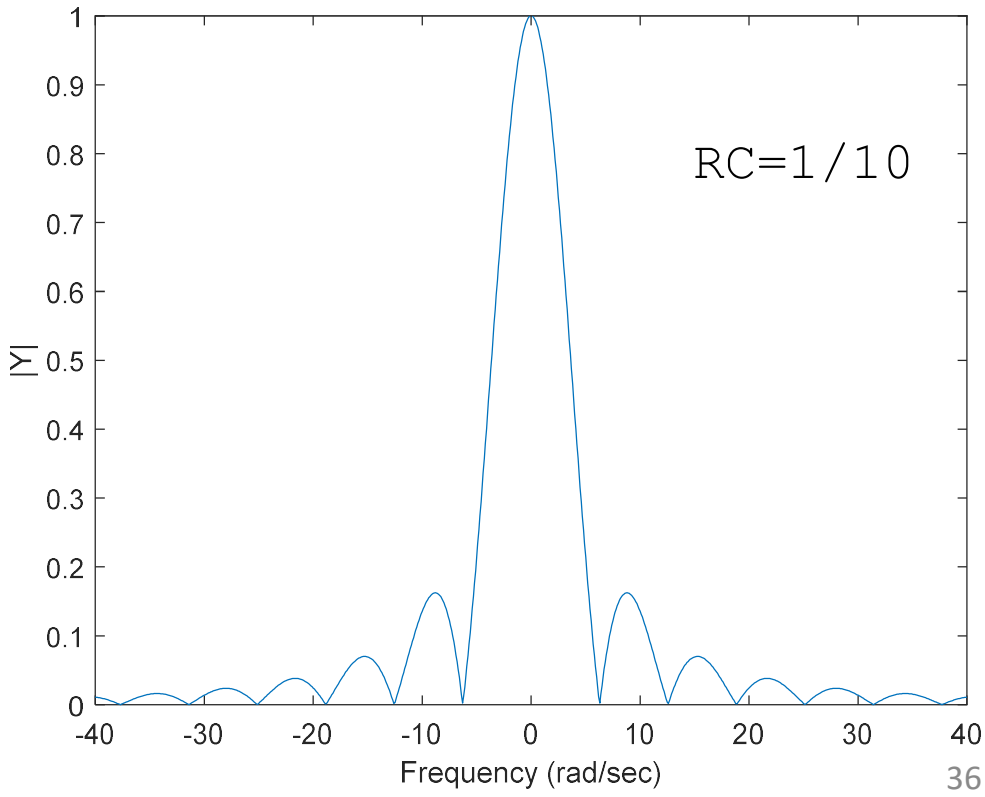
where

$$H(\omega) = \frac{1/RC}{j\omega + 1/RC}$$

$$|Y(\omega)| = |H(\omega)||X(\omega)|$$

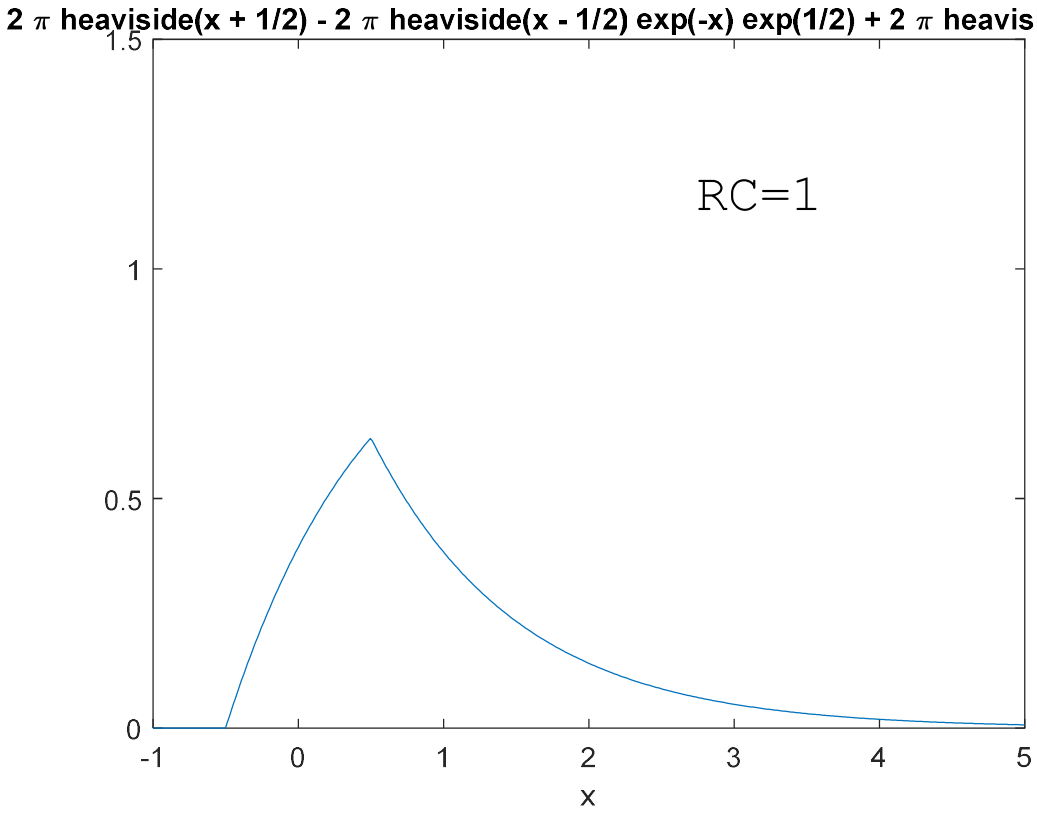
```
RC=1;
w=-40:0.1:40;
X=2*sin(w/2)./w;
H=(1/RC)./(j*w+1/RC);
Y=X.*H;
plot(w,abs(Y))
xlabel('Frequency (rad/sec)')
ylabel('|Y|')
```

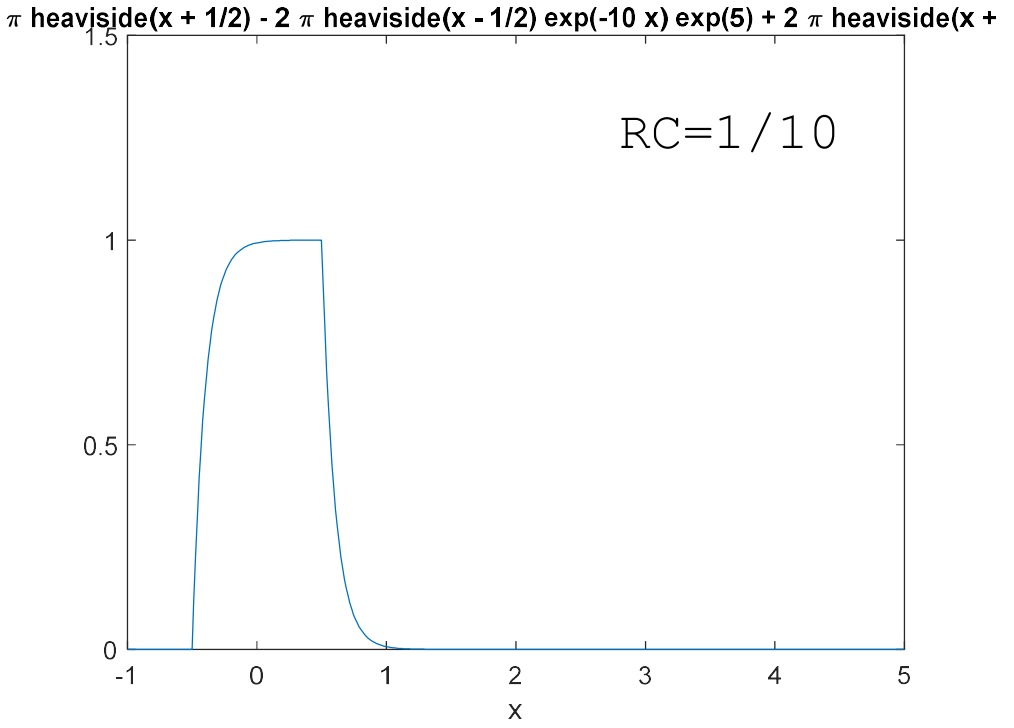

RC=1
|Y|
Frequency (rad/sec)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-40
-30
-20
-10
0
10
20
30
40


RC=1/10
|Y|
Frequency (rad/sec)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-40
-30
-20
-10
0
10
20
30
40

Compute y(t)

```
syms X H Y y w
RC=1;
X=2*sin(w/2)./w;
H=(1/RC)./(j*w+1/RC);
Y=X.*H;
y=ifourier(Y);
ezplot(y,[-1,5])
axis([-1 5 0 1.5])
```


2 π heaviside(x + 1/2) - 2 π heaviside(x - 1/2) exp(-x) exp(1/2) + 2 π heaviside(x +
RC=1
1.5
1
0.5
0
-1
0
1
2
3
4
5
x


π heaviside(x + 1/2) - 2 π heaviside(x - 1/2) exp(-10 x) exp(5) + 2 π heaviside(x +
RC=1/10
1.5
1
0.5
0
-1
0
1
2
3
4
5
x