EE 362: Signals and Systems

Properties of DTFT, DFT, Inverse DFT, Fast Fourier Transform

Lecture 11

Properties of Discrete-time Fourier Transform (DTFT)

Linearity: DTFT is a linear operation.

If: $\begin{cases} x[n] \overset{\Im}{\longleftrightarrow} X(\Omega) \\ v[n] \overset{\Im}{\longleftrightarrow} V(\Omega) \end{cases}$ Then: $ax[n]+bv[n] \overset{\Im}{\longleftrightarrow} aX(\Omega)+bV(\Omega)$

a,*b* are constant:

Left/right shift in time: *a* is a positive/negative integer number

If: $x[n] \overset{\Im}{\longleftrightarrow} X(\Omega)$ Then: $x[n-a] \overset{\Im}{\longleftrightarrow} X(\Omega)e^{-j\Omega a}$

Time Reversal:

If: $x[n] \overset{\Im}{\longleftrightarrow} X(\Omega)$ Then: $x[-n] \overset{\Im}{\longleftrightarrow} X(-\Omega) = X(\Omega)^*$

Proof:

$$x[-n] \overset{\Im}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} x[-n]e^{-j\Omega n} = \sum_{k=\infty}^{-\infty} x[k]e^{-j\Omega(-k)} = \sum_{k=-\infty}^{\infty} x[k]e^{-j(-\Omega)k} = X(-\Omega) = X(\Omega)^*$$

Multiplication by *n*:

If: $x[n] \overset{\mathfrak{I}}{\longleftrightarrow} X(\Omega)$ Then: $nx[n] \overset{\mathfrak{I}}{\longleftrightarrow} j\frac{d}{d\Omega}X(\Omega)$

Proof:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \Rightarrow \frac{dX(\Omega)}{d\Omega} = -j\sum_{n=-\infty}^{\infty} nx[n]e^{-j\Omega n} \Rightarrow j\frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} nx[n]e^{-j\Omega n}$$

Multiplying by a complex exponential:

If: $x[n] \overset{\mathfrak{I}}{\longleftrightarrow} X(\Omega)$ Then: $x[n]e^{j\Omega_0 n} \overset{\mathfrak{I}}{\longleftrightarrow} X(\Omega - \Omega_0)$

Proof:

$$\sum_{n=-\infty}^{\infty} x[n]e^{j\Omega_0 n}e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\Omega-\Omega_0)n} = X(\Omega - \Omega_0)$$

Multiplying by a sinusoidal :

If: $x[n] \overset{\mathfrak{I}}{\longleftrightarrow} X(\Omega)$ Then:

$$\begin{cases} x[n]\cos(\Omega_0 n) \overset{\mathfrak{I}}{\longleftrightarrow} \frac{1}{2}\{X(\Omega-\Omega_0)+X(\Omega+\Omega_0)\} \\ x[n]\sin(\Omega_0 n) \overset{\mathfrak{I}}{\longleftrightarrow} \frac{j}{2}\{X(\Omega+\Omega_0)-X(\Omega-\Omega_0)\} \end{cases}$$

Convolution in the time domain:

If: $\begin{cases} x[n] \xleftrightarrow{\Im} X(\Omega) \\ v[n] \xleftrightarrow{\Im} V(\Omega) \end{cases}$ Then: $x[n]*v[n] \xleftrightarrow{\Im} X(\Omega)V(\Omega)$

Proof: using the definition of convolution

$$f\{x[n]*v[n]\} = \sum_{n=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} x[i]v[n-i]e^{-j\Omega n} \xrightarrow{n-i=k} \sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} x[i]v[k]e^{-j\Omega(i+k)} =$$

$$= \sum_{i=-\infty}^{\infty} x[i]e^{-j\Omega(i)} \sum_{k=-\infty}^{\infty} v[k]e^{-j\Omega(k)} = X(\Omega)V(\Omega)$$

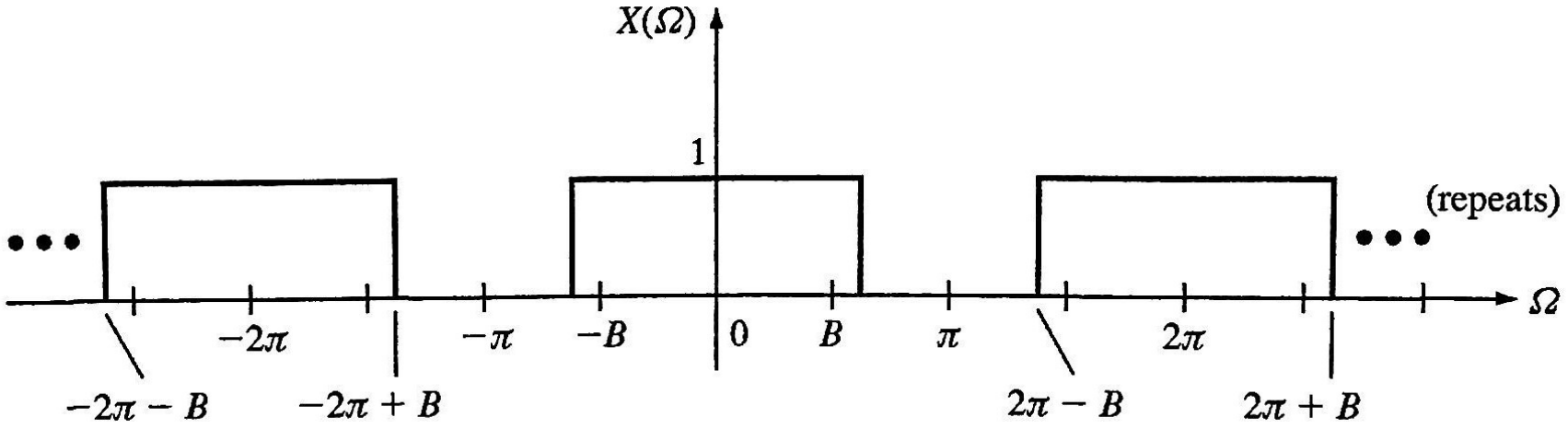
Summation:

If: $x[n] \xleftrightarrow{\Im} X(\Omega)$ Then: $\sum_{i=0}^{n} x[i] \xleftrightarrow{\Im} \frac{1}{1-e^{-j\Omega}} X(\Omega) + \pi \sum_{n=-\infty}^{\infty} X(2\pi n)\delta(\Omega - 2\pi n)$

TABLE 4.2 Properties of the DTFT

Property	Transform Pair/Property
Linearity	$ax[n] + bv[n] \leftrightarrow aX(\Omega) + bV(\Omega)$
Right or left shift in time	$x[n - q] \leftrightarrow X(\Omega)e^{-jq\Omega}$, q any integer
Time reversal	$x[-n] \leftrightarrow X(-\Omega) = \overline{X(\Omega)}$
Multiplication by n	$nx[n] \leftrightarrow j\frac{d}{d\Omega}X(\Omega)$
Multiplication by a complex exponential	$x[n]e^{jn\Omega_0} \leftrightarrow X(\Omega - \Omega_0)$, Ω_0 real
Multiplication by $\sin \Omega_0 n$	$x[n] \sin \Omega_0 n \leftrightarrow \frac{j}{2}[X(\Omega + \Omega_0) - X(\Omega - \Omega_0)]$
Multiplication by $\cos \Omega_0 n$	$x[n] \cos \Omega_0 n \leftrightarrow \frac{1}{2}[X(\Omega + \Omega_0) + X(\Omega - \Omega_0)]$
Convolution in the time domain	$x[n] * v[n] \leftrightarrow X(\Omega)V(\Omega)$
Summation	$\sum_{i=0}^{n} x[i] \leftrightarrow \frac{1}{1 - e^{-j\Omega}}X(\Omega) + \sum_{n=-\infty}^{\infty} \pi X(2\pi n)\delta(\Omega - 2\pi n)$
Multiplication in the time domain	$x[n]v[n] \leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega - \lambda)V(\lambda)\, d\lambda$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x[n]v[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{X(\Omega)}V(\Omega)\, d\Omega$
Special case of Parseval's theorem	$\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(\Omega)\rvert^2\, d\Omega$
Relationship to inverse CTFT	If $x[n] \leftrightarrow X(\Omega)$ and $\gamma(t) \leftrightarrow X(\omega)p_{2\pi}(\omega)$, then $x[n] = \gamma(t)\lvert_{t=n} = \gamma(n)$

Example: Suppose that $X(\Omega) = \sum_{k=-\infty}^{\infty} p_{2B}(\Omega + 2\pi k)$ where $B < \pi$



It is seen that $X(\omega)p_{2\pi}(\omega) = p_{2B}(\omega)$

the inverse CTFT of $p_{2B}(\omega)$ is equal to $\frac{B}{\pi}\text{sinc}\left(\frac{B}{\pi}t\right), \quad -\infty < t < \infty$

Thus, $$x[n] = \gamma[n] = \frac{B}{\pi}\text{sinc}\left(\frac{B}{\pi}n\right), \quad n = 0, \pm 1, \pm 2, \ldots$$

which yields the DTFT pair

$$\frac{B}{\pi}\text{sinc}\left(\frac{B}{\pi}n\right) \leftrightarrow \sum_{k=-\infty}^{\infty} p_{2B}(\Omega + 2\pi k)$$

Discrete Fourier Transform (DFT): Since $X(\Omega)$ is a continuous function of Ω, it cannot be stored in a digital computer unless it can be discretized and expressed in a closed form. This leads to the concept of Discrete Fourier Transform (DFT):

$$X_k = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \qquad k = 0,1,2,\ldots,N-1$$

DFT function using MATLAB

```
%%%
```

Example: Find the Discrete Fourier Transform (DFT) of $x[n]$ defined by $x[0]=1$, $x[1]=2$, $x[2]=2$, $x[3]=1$, and $x[n]=0$ for other n, over $n=0$, $n=N-1=3$

$$X_k = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi kn}{4}} \qquad k = 0,1,2,3$$

$$X_k = x[0] + x[1] e^{-j\frac{2\pi k}{4}} + x[2] e^{-j\frac{4\pi k}{4}} + x[3] e^{-j\frac{6\pi k}{4}}$$

$$= 1 + 2e^{-j\frac{\pi k}{2}} + 2e^{-j\frac{\pi k}{1}} + 1e^{-j\frac{3\pi k}{2}}$$

$$X_k = \begin{cases} 6, & k = 0 \\ -1 - j, & k = 1 \\ 0, & k = 2 \\ -1 + j, & k = 3 \end{cases}$$

MATLAB:

```
x= [1  2  2  1];
Xk =dft(x)
```

```
>> x= [1  2  2  1];
Xk =dft(x)

Xk =

   6.0000 + 0.0000i
  -1.0000 - 1.0000i
   0.0000 - 0.0000i
  -1.0000 + 1.0000i
```

Inverse Discrete Fourier Transform (IDFT): If X_k is the N-point DFT of $x[n]$, then $x[n]$ can be determined from X_k by

$$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X_k e^{j\frac{2\pi kn}{N}} \qquad n = 0,1,2,\ldots,N-1$$

```
% Inverse DFT
function x=idft(Xk)
[N,M]=size(Xk);
if M~=1 % makes sure that x is a column vector
    Xk=Xk.'; % takes the transpose without
conjugation
    N=M;
end
x=zeros(N,1);
k=0:N-1;
for n=0:N-1;
    x(n+1)=exp(1i*2*pi*k*n/N)*Xk;
end
x=x/N;
end
```

Example: Find the Inverse Discrete Fourier Transform (IDFT) of X_k defined by

$$X_k = \begin{cases} 6, & k = 0 \\ -1 - j, & k = 1 \\ 0, & k = 2 \\ -1 + j, & k = 3 \end{cases}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi kn}{N}} = \frac{1}{4}[X_0 + X_1 e^{j\frac{2\pi n}{4}} + X_2 e^{j\frac{4\pi n}{4}} + X_3 e^{j\frac{6\pi n}{4}}] \qquad n = 0,1,2,3$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi kn}{N}} = \frac{1}{4}[X_0 + X_1 e^{j\frac{2\pi n}{4}} + X_2 e^{j\frac{4\pi n}{4}} + X_3 e^{j\frac{6\pi n}{4}}] \qquad n = 0,1,2,3$$

$$x[0] = \frac{1}{4}[X_0 + X_1 + X_2 + X_3] = 1$$

$$x[1] = \frac{1}{4}[X_0 + X_1 e^{j\frac{2\pi}{4}} + X_2 e^{j\frac{4\pi}{4}} + X_3 e^{j\frac{6\pi}{4}}] = 2$$

$$x[2] = \frac{1}{4}[X_0 + X_1 e^{j\frac{4\pi}{4}} + X_2 e^{j\frac{8\pi}{4}} + X_3 e^{j\frac{12\pi}{4}}] = 2$$

$$x[3] = \frac{1}{4}[X_0 + X_1 e^{j\frac{6\pi}{4}} + X_2 e^{j\frac{12\pi}{4}} + X_3 e^{j\frac{18\pi}{4}}] = 1$$

MATLAB:

Xk = [6 -1-j 0 -1+j]; % define X_k

x=idft(Xk) % compute IDFT

```
>> Xk=[6 -1-j 0 -1+j];
>> x=idft(Xk)

x =

   1.0000 + 0.0000i
   2.0000 - 0.0000i
   2.0000 - 0.0000i
   1.0000 + 0.0000i
```

Fast Fourier Transform (FFT) Algorithm

Given a D-T signal $x[n]$, the N-point DFT and inverse DFT were defined

$$X_k = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \qquad k = 0,1,2,\ldots,N-1$$

$$x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X_k e^{j\frac{2\pi kn}{N}} \qquad n = 0,1,2,\ldots,N-1$$

The direct computation of X_k and $x[n]$ require N^2 multiplications each. Using FFT it requires only $(N\ log_2 N)/2$ multiplications. If $N = 1024$, direct computation requires $N^2 = 1{,}048{,}576$ multiplications and FFT requires 5120 multiplications.

MATLAB

x = [1 2 2 1]; % define x

Xk=fft(x)

```
>> x = [1  2 2 1]; % define x
Xk=fft(x)

Xk =

   6.0000 + 0.0000i  -1.0000 - 1.0000i   0.0000 + 0.0000i  -1.0000 + 1.0000i
```

Xk = [6 -1-j 0 -1+j]; % define Xk

x=ifft(Xk) % inverse fft

```
>> Xk = [6  -1-j 0 -1+j]; % define Xk
x=ifft(Xk) % inverse fft

x =

     1     2     2     1
```

Examples

Fourier Series	$x[n] = \sum_{k=<N>} a_k e^{jk\Omega_0 n}$ $a_k = \frac{1}{N} \sum_{n=<N>} x[n]\, e^{-jk\Omega_0 n}$
Discrete-Time Fourier Transform	$X(\Omega) = \sum_{n=-\infty}^{n=\infty} x[n] e^{-j\Omega n}$
Inverse DTFT	$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X(\Omega) e^{j\Omega n} d\Omega$
Discrete Fourier Transform (DFT)	$X_k = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \qquad k = 0,1,2,\ldots,N-1$
Inverse DFT	$x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X_k e^{j\frac{2\pi kn}{N}} \qquad n = 0,1,2,\ldots,N-1$

TABLE 4.1 Common DTFT Pairs

$$1, \text{ all } n \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega - 2\pi k)$$

$$\text{sgn}[n] \leftrightarrow \frac{2}{1 - e^{-j\Omega}}, \text{ where } \text{sgn}[n] = \begin{cases} 1, & n = 0, 1, 2, \ldots \\ -1, & n = -1, -2, \ldots \end{cases}$$

$$u[n] \leftrightarrow \frac{1}{1 - e^{-j\Omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\Omega - 2\pi k)$$

$$\delta[n] \leftrightarrow 1$$

$$\delta[n - q] \leftrightarrow e^{-jq\Omega}, q = \pm 1, \pm 2, \ldots$$

$$a^n u[n] \leftrightarrow \frac{1}{1 - ae^{-j\Omega}}, \quad |a| < 1$$

$$e^{j\Omega_0 n} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0 - 2\pi k)$$

$$p[n] \leftrightarrow \frac{\sin\left[\left(q + \frac{1}{2}\right)\Omega\right]}{\sin(\Omega/2)}$$

$$\frac{B}{\pi}\text{sinc}\left(\frac{B}{\pi}n\right) \leftrightarrow \sum_{k=-\infty}^{\infty} p_{2B}(\Omega + 2\pi k)$$

$$\cos \Omega_0 n \leftrightarrow \sum_{k=-\infty}^{\infty} \pi[\delta(\Omega + \Omega_0 - 2\pi k) + \delta(\Omega - \Omega_0 - 2\pi k)]$$

$$\sin \Omega_0 n \leftrightarrow \sum_{k=-\infty}^{\infty} j\pi[\delta(\Omega + \Omega_0 - 2\pi k) - \delta(\Omega - \Omega_0 - 2\pi k)]$$

$$\cos(\Omega_0 n + \theta) \leftrightarrow \sum_{k=-\infty}^{\infty} \pi[e^{-j\theta}\delta(\Omega + \Omega_0 - 2\pi k) + e^{j\theta}\delta(\Omega - \Omega_0 - 2\pi k)]$$

TABLE 4.2 Properties of the DTFT

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Multiplication by $\sin \Omega_0 n$	$x[n] \sin \Omega_0 n \leftrightarrow \frac{j}{2}[X(\Omega + \Omega_0) - X(\Omega - \Omega_0)]$
Multiplication by $\cos \Omega_0 n$	$x[n] \cos \Omega_0 n \leftrightarrow \frac{1}{2}[X(\Omega + \Omega_0) + X(\Omega - \Omega_0)]$
Convolution in the time domain	$x[n] * v[n] \leftrightarrow X(\Omega)V(\Omega)$
Summation	$\sum_{i=0}^{n} x[i] \leftrightarrow \frac{1}{1 - e^{-j\Omega}}X(\Omega) + \sum_{n=-\infty}^{\infty} \pi X(2\pi n)\delta(\Omega - 2\pi n)$
Multiplication in the time domain	$x[n]v[n] \leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega - \lambda)V(\lambda)\, d\lambda$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x[n]v[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{X(\Omega)}V(\Omega)\, d\Omega$
Special case of Parseval's theorem	$\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(\Omega)\rvert^2\, d\Omega$
Relationship to inverse CTFT	If $x[n] \leftrightarrow X(\Omega)$ and $\gamma(t) \leftrightarrow X(\omega)p_{2\pi}(\omega)$, then $x[n] = \gamma(t)\lvert_{t=n} = \gamma(n)$

DFT function using MATLAB

```
%%%
```

```
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conjugation
    N=M;
end
x=zeros(N,1);
k=0:N-1;
for n=0:N-1;
    x(n+1)=exp(1i*2*pi*k*n/N)*Xk;
end
x=x/N;
end
```

Examples

- Example 1: Compute the DTFT of the following D-T signal.

$x[-2] = 2, x[-1] = 2, x[0] = 2, x[1] = 2,$
$x[2] = 2, x[3] = 3, x[4] = 3, x[5] = 3,$ and
$x[n] = 0$ for other n.

Example 2: Compute the DFT of the following D-T signal.

$x[-2] = 2, x[-1] = 2, x[0] = 2, x[1] = 2,$
$x[2] = 2, x[3] = 3, x[4] = 3, x[5] = 3,$ and
$x[n] = 0$ for other n.

Example 3: Compute the inverse DTFT of the following signal.

