EE 362: Signals and Systems

Discrete-Time Fourier Transform (DTFT), Inverse DTFT, Generalized DTFT

Lecture 10

Discrete-Time Fourier Transform (DTFT)

As is the case for the Fourier transform of a C-T signal, the DTFT of a D-T signal is a function of a **continuum** of frequencies. But, the DTFT is always a periodic function with period 2π.

Given a discrete-time signal $x[n]$, the discrete-time Fourier transform (DTFT) of $x[n]$ is defined:

$$X(\Omega) = \sum_{n=-\infty}^{n=\infty} x[n] e^{-j\Omega n}$$

$x[n]$ should be absolutely summable:

$$\sum_{n=-\infty}^{n=\infty} |x[n]| < \infty$$

For any D-T signal $x[n]$, the DTFT $X[\Omega]$ is a periodic function of Ω with period 2π, that is

$$X[\Omega + 2\pi] = X[\Omega] \; for \; all \; \Omega, -\infty < \Omega < \infty$$

To prove the periodicity,

$$X[\Omega + 2\pi] = \sum_{n=-\infty}^{\infty} x[n] e^{-jn(\Omega+2\pi)}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega} e^{-jn2\pi}$$

But, $e^{-jn2\pi} = 1$, thus

$$\boldsymbol{X[\Omega + 2\pi] = X[\Omega] \; for \; all \; \Omega, -\infty < \Omega < \infty}$$

- An important consequence of periodicity of $X[\Omega]$ is that $X[\Omega]$ is completely determined by the computation of $X[\Omega]$ over any 2π interval such as $0 \leq \Omega \leq 2\pi$ or $-\pi \leq \Omega \leq \pi$

The discrete-time Fourier transform $X(\Omega)$ is a complex value, so it can be expressed in either rectangular or polar form:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \xleftrightarrow{\text{using Euler's formula}} \underbrace{\sum_{n=-\infty}^{\infty} x[n]\cos(\Omega n)}_{R(\Omega)} + j\underbrace{(-\sum_{n=-\infty}^{\infty} x[n]\sin(\Omega n))}_{jI(\Omega)}$$

$$X(\Omega) = R(\Omega) + jI(\Omega) = |X(\Omega)| e^{j\angle X(\Omega)}$$

$$R(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\cos(\Omega n)$$ Real part of $X(\Omega)$

$$I(\Omega) = -\sum_{n=-\infty}^{\infty} x[n]\sin(\Omega n)$$ Imaginary part of $X(\Omega)$

Even or odd symmetry:

$$R(\Omega) = \begin{cases} x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\Omega n) & \text{when } x[n] \text{ is even} \\ 0 \quad \text{when } x[n] \text{ is odd} \end{cases}$$

$$I(\Omega) = \begin{cases} x[0] \quad \text{when } x[n] \text{ is even} \\ -2\sum_{n=1}^{\infty} x[n]\sin(\Omega n) \quad \text{when } x[n] \text{ is odd} \end{cases}$$

Spectrum of a D-T Signal

- For a D-T signal $x[n]$ that is not equal to a sum of sinusoids, the frequency spectrum consists of a continuum of frequency components that make up the signal.
- The magnitude function $|X(\Omega)|$ is called amplitude spectrum of the signal.
- The angle function $\angle X(\Omega)$ is called the phase spectrum of the signal.

Inverse DTFT: the inverse discrete-time Fourier transform of $X(\Omega)$ is defined by:

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$

$$X(\Omega + 2\pi) = X(\Omega)$$

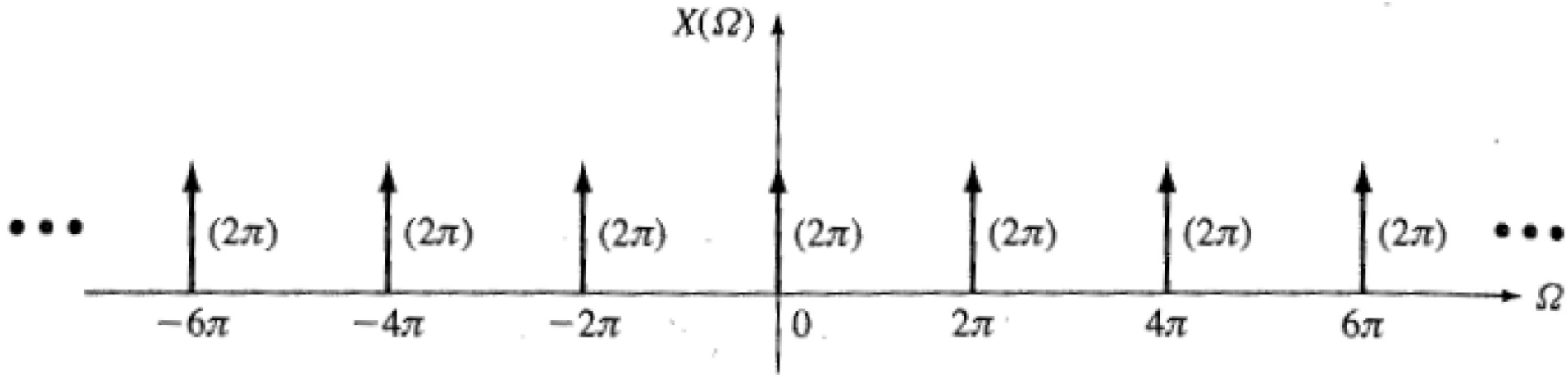
Since $X(\Omega)$ is periodic

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega$$

Generalized DTFT: Some signals do not have a DTFT in the ordinary sense. E.g., the constant signal $x[n] = 1$ for all integers n does not have a DTFT.

But it has a generalized DTFT which is defined by the impulse train

$$X(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega - 2k\pi)$$



Proof: $$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\Omega)e^{j\Omega n}d\Omega=\frac{1}{2\pi}\int_{-\pi}^{\pi}[\sum_{k=-\infty}^{\infty}2\pi\delta(\Omega-2k\pi)]e^{j\Omega n}d\Omega=$$

$$\int_{-\pi}^{\pi}[\sum_{k=-\infty}^{\infty}\delta(\Omega-2k\pi)]e^{j(2k\pi)n}d\Omega=1$$

Note: Since $X(\Omega)$ is periodic, the integral can be taken over any full period. That is, the reason we have a impulse train (each pulse is shifted by 2π)

Transform Pair:

$$x[n]\overset{\mathfrak{I}}{\longleftrightarrow}X(\Omega)$$

$$\begin{cases} X(\Omega)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n} & \text{DTFT} \\ x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\Omega)e^{j\Omega n}d\Omega & \text{Inverse DTFT} \end{cases}$$

TABLE 4.1 Common DTFT Pairs

$$1, \text{all } n \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega - 2\pi k)$$

$$\text{sgn}[n] \leftrightarrow \frac{2}{1 - e^{-j\Omega}}, \text{ where } \text{sgn}[n] = \begin{cases} 1, & n = 0, 1, 2, \ldots \\ -1, & n = -1, -2, \ldots \end{cases}$$

$$u[n] \leftrightarrow \frac{1}{1 - e^{-j\Omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\Omega - 2\pi k)$$

$$\delta[n] \leftrightarrow 1$$

$$\delta[n - q] \leftrightarrow e^{-jq\Omega}, q = \pm 1, \pm 2, \ldots$$

$$a^n u[n] \leftrightarrow \frac{1}{1 - ae^{-j\Omega}}, \quad |a| < 1$$

$$e^{j\Omega_0 n} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0 - 2\pi k)$$

$$p[n] \leftrightarrow \frac{\sin\left[\left(q + \frac{1}{2}\right)\Omega\right]}{\sin(\Omega/2)}$$

$$\frac{B}{\pi}\text{sinc}\left(\frac{B}{\pi}n\right) \leftrightarrow \sum_{k=-\infty}^{\infty} p_{2B}(\Omega + 2\pi k)$$

$$\cos \Omega_0 n \leftrightarrow \sum_{k=-\infty}^{\infty} \pi[\delta(\Omega + \Omega_0 - 2\pi k) + \delta(\Omega - \Omega_0 - 2\pi k)]$$

$$\sin \Omega_0 n \leftrightarrow \sum_{k=-\infty}^{\infty} j\pi[\delta(\Omega + \Omega_0 - 2\pi k) - \delta(\Omega - \Omega_0 - 2\pi k)]$$

$$\cos(\Omega_0 n + \theta) \leftrightarrow \sum_{k=-\infty}^{\infty} \pi[e^{-j\theta}\delta(\Omega + \Omega_0 - 2\pi k) + e^{j\theta}\delta(\Omega - \Omega_0 - 2\pi k)]$$

Example:

$$x[n] \overset{\mathfrak{I}}{\longleftrightarrow} X(\Omega)$$

$$\delta[n] \overset{\mathfrak{I}}{\longleftrightarrow} 1 \qquad X(\Omega) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\Omega n} = \delta[0] e^{-j\Omega(0)} = 1$$

$$\delta[n-p] \overset{\mathfrak{I}}{\longleftrightarrow} e^{-j\Omega p}$$

Example: Compute the discrete Fourier transform of $x[n]$

$$x[n] = \begin{cases} 0, & n < 0, n > p \\ a^n, & 0 \le n \le p \end{cases}$$

where a is a nonzero real constant and p is a positive integer

$$X(\Omega) = \sum_{n=-\infty}^{n=\infty} x[n]e^{-j\Omega n} = \sum_{n=0}^{n=p} a^n e^{-j\Omega n} = \sum_{n=0}^{n=p} (ae^{-j\Omega})^n = \frac{1-(ae^{-j\Omega})^{p+1}}{1-ae^{-j\Omega}}$$

Remember:

$$\sum_{i=0}^{n} a^i = \frac{1-a^{n+1}}{1-a}$$

Example: Compute the discrete Fourier transform of $x[n]$

$$x[n] = a^n u[n], \qquad |a| < 1$$

$$X(\Omega) = \sum_{n=-\infty}^{n=\infty} x[n]u[n]e^{-j\Omega n} = \sum_{n=0}^{\infty} a^n e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\Omega})^n = \frac{1-(ae^{-j\Omega})^{\infty}}{1-ae^{-j\Omega}} = \frac{1}{1-ae^{-j\Omega}}$$

Example: Compute the discrete time Fourier transform (DTFT) of the rectangular pulse

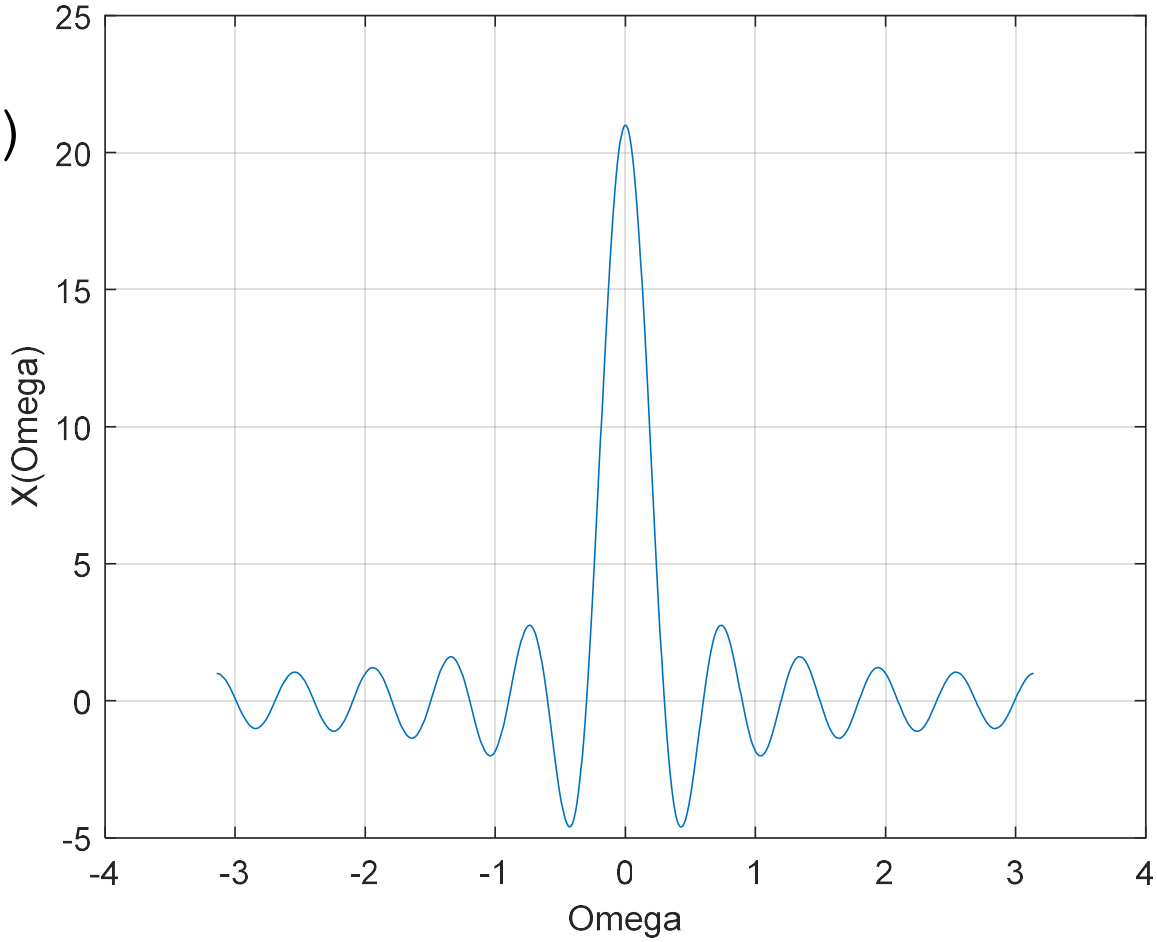
$$x[n] = \begin{cases} 1, & n = -q, -q+1, \ldots, q-1, q \\ 0, & \text{o.w} \end{cases}$$

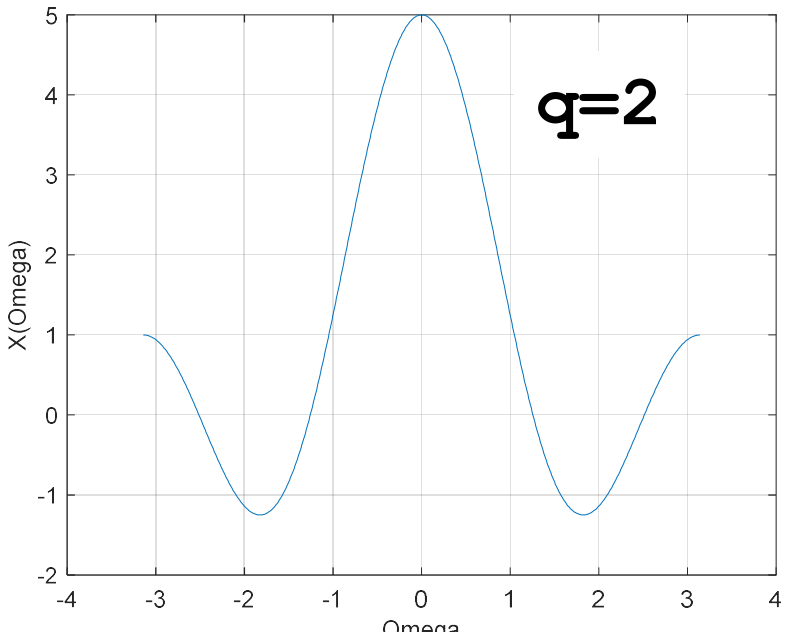
$$X(\Omega) = \sum_{n=-\infty}^{n=\infty} x[n] e^{-j\Omega n} = \sum_{n=-q}^{q} e^{-j\Omega n} \xrightarrow{k=n+q} \sum_{k=0}^{2q} (e^{-j\Omega})^{k-q} = e^{jq\Omega} \sum_{k=0}^{2q} (e^{-j\Omega})^{k}$$

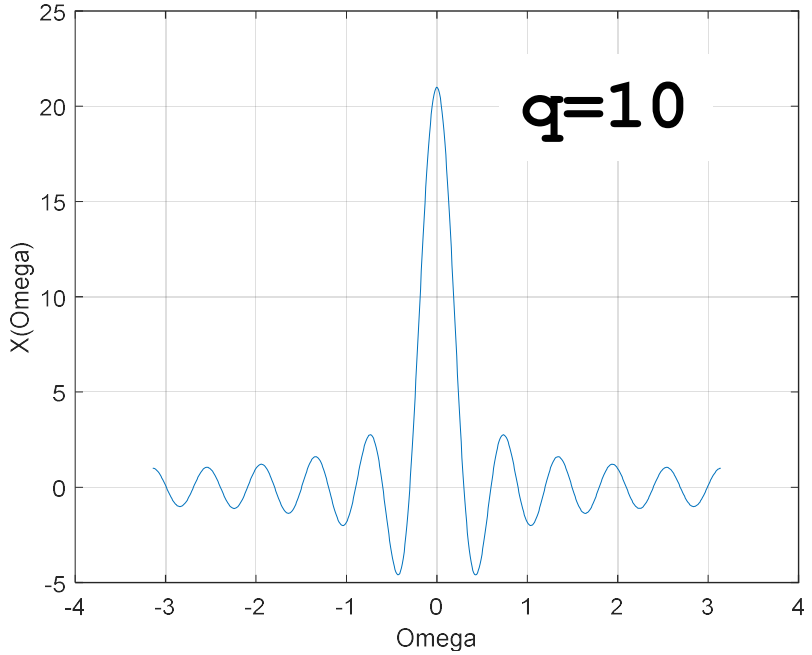
$$= e^{jq\Omega} \frac{1-(e^{-j\Omega})^{2q+1}}{1-e^{-j\Omega}} = \frac{e^{jq\Omega} - e^{-j(q+1)\Omega}}{1-e^{-j\Omega}}$$

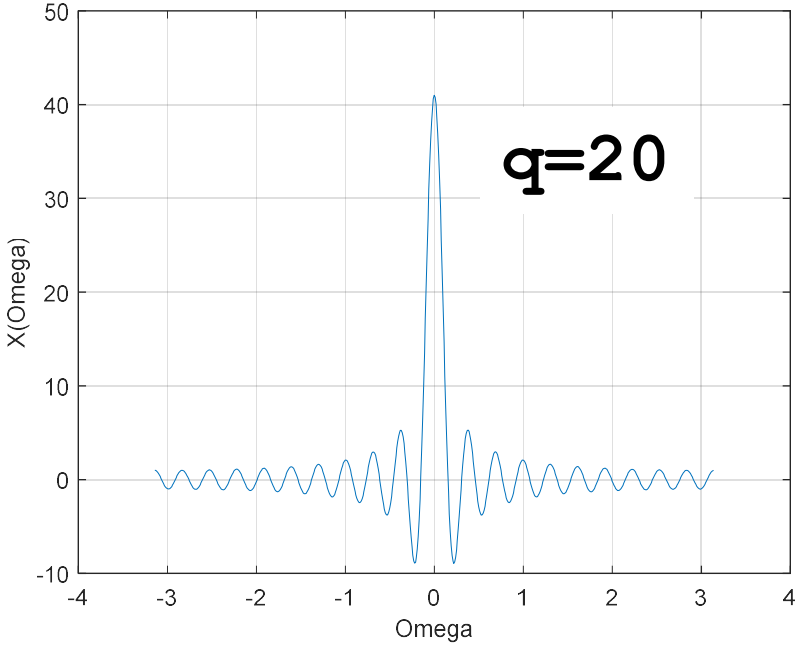
$$= \frac{e^{j(q+\frac{1}{2})\Omega} - e^{-j(q+\frac{1}{2})\Omega}}{e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}} = \frac{\sin((q+\frac{1}{2})\Omega)}{\sin(\frac{\Omega}{2})}$$

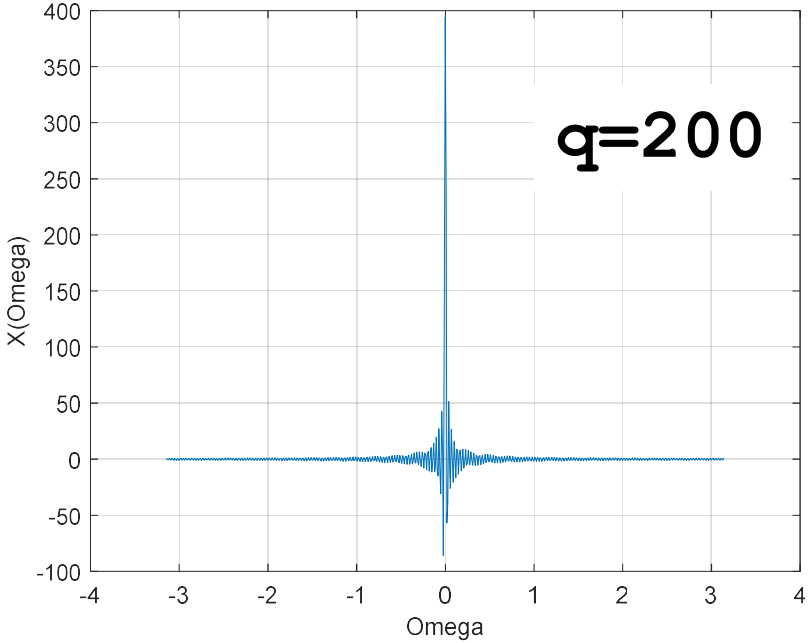
```
OM=-pi:0.01:pi;
q=10;
X=sin((q+0.5)*OM)./sin(0.5*OM);
plot(OM,X)
xlabel('Omega')
ylabel('X(Omega)')
grid
```




q=2
X(Omega)
Omega


q=10
X(Omega)
Omega


q=20
X(Omega)
Omega


q=200
X(Omega)
Omega