EE 362: Signals and Systems

Fourier Analysis of Discrete-Time Signals

Lecture 9

Fourier Analysis of Discrete-Time Signals

- Discrete-time signals are decomposed into a linear combination of complex exponentials.
- For periodic signals: discrete-time Fourier series
- For aperiodic signals: discrete-time Fourier transform (DTFT).
- Since the frequency (Ω) is a continuous variable, to implement DTFT on a digital computer, it is necessary to discretize in frequency. This leads to, **Discrete Fourier Transform (DFT)**
- We will also study the Fast Fourier Transform (**FFT**)

Fourier Series of Periodic DT Signals

- The signal $x[n]$ is periodic with a period N
- The fundamental frequency is $\Omega_0 = \frac{2\pi}{N}$
- The signal $x[n]$ can be decomposed into:

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n}$$

Because $x[n]$ is periodic, $e^{jk\Omega_0 n} = e^{j(k+N)\Omega_0 n}$

In other words, when $k = N$ it produces the same complex exponential when $k = 0$

Then, $k = 0,1,2,\ldots,N-1$, or $k = 1,2,\ldots,N$, or $k = 2,\ldots,N+1$, and so on

Then,

$$x[n] = \sum_{k=<N>} a_k e^{jk\Omega_0 n}$$

where $k =< N >$ means k ranges for one period.

In general, we will have

N equations in N unknowns

The coefficient a_k can be expressed as follows:

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n]\, e^{-jk\Omega_0 n}$$

Consider the partial sum of $x[n]$

$$\hat{x}[n] = \frac{1}{N} \sum_{P\ terms} a_k e^{jk\Omega_0 n}$$

when $P = N$ we get,

$$\hat{x}[n] = x[n]$$

Then D-T Fourier series does not have convergence issues such as those required for C-T (e.g. square integrable & Dirichlet conditions)

- Example: Periodic discrete-time square wave with period $N = 4$:

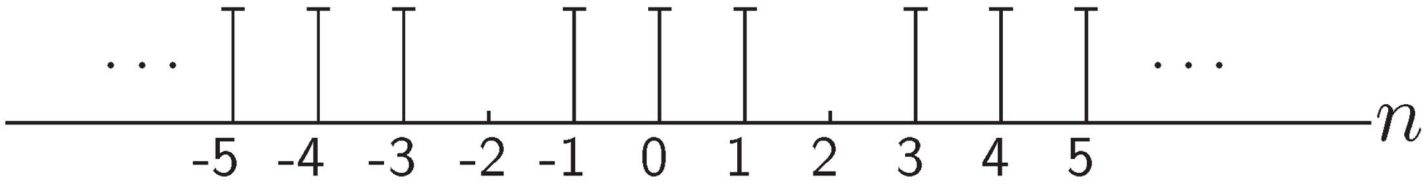
Even signal

$x[0] = 1$

$x[1] = 1$

$x[2] = 0$

$x[3] = 1$



$$k = 0, 1, 2, 3$$

$\Omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$, then frequency components are:

$0, \frac{1(\pi)}{2}, \frac{2(\pi)}{2}, \frac{3(\pi)}{2}$

The coefficient a_k

$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk\Omega_0 n}$$

$$= \frac{1}{4}\left[1 \,.\, e^{0} + 1 \,.\, e^{-jk\Omega_0} + 0 + 1 \,.\, e^{-j3k\Omega_0}\right]$$

$$= \frac{1}{4}\left[1 + e^{-jk\Omega_0} + e^{-j3k\Omega_0}\right]$$

$$= \frac{1}{4}[1 + \cos(k\Omega_0) - j\,\sin(k\Omega_0) + \cos(3k\Omega_0) - j\,\sin(3k\Omega_0)]$$

$$= \frac{1}{4}\left[1 + \cos\left(k\frac{\pi}{2}\right) - j\,\sin(k\frac{\pi}{2}) + \cos\left(3k\frac{\pi}{2}\right) - j\,\sin(3k\frac{\pi}{2})\right]$$

Since this signal is even, the sine term should vanish

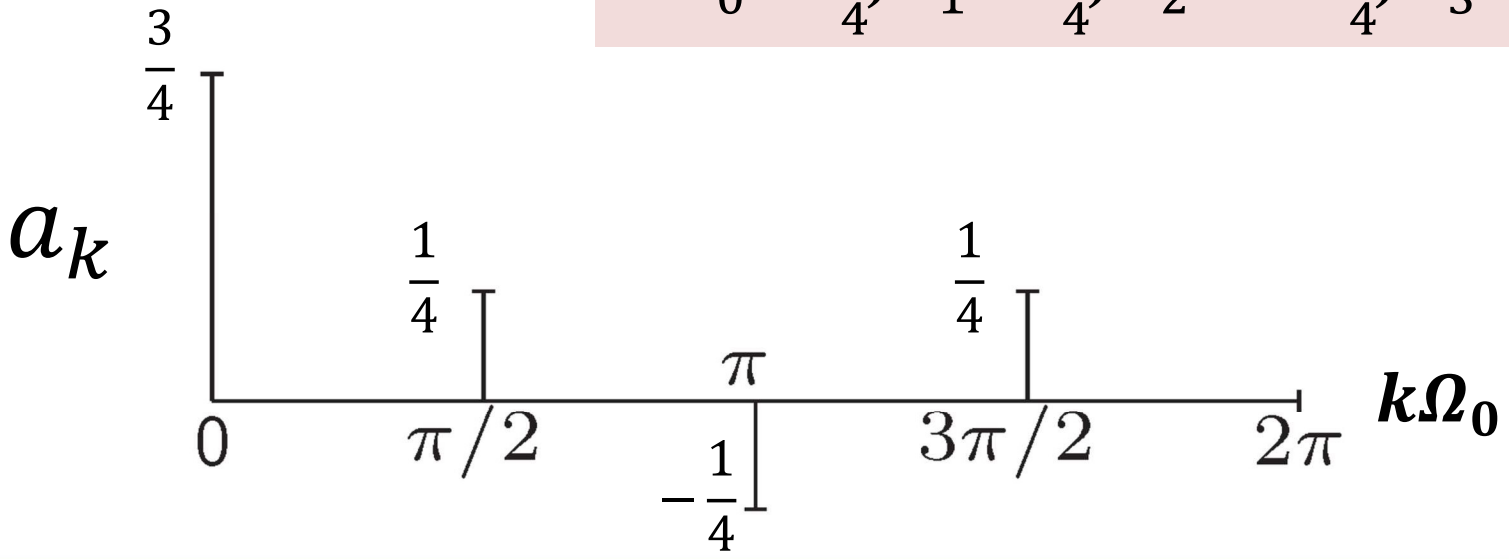
$$-\sin\left(k\frac{\pi}{2}\right) = -\sin\left(k\frac{\pi}{2} - 2\pi k\right) = -\sin\left(-3k\frac{\pi}{2}\right) = \sin\left(3k\frac{\pi}{2}\right)$$

$$\cos\left(3k\frac{\pi}{2}\right) = \cos\left(3k\frac{\pi}{2} - 2\pi k\right) = \cos\left(-k\frac{\pi}{2}\right) = \cos\left(k\frac{\pi}{2}\right)$$

$$a_k = \frac{1}{4}\left[1 + \cos\left(k\frac{\pi}{2}\right) - j\sin(k\frac{\pi}{2}) + \cos\left(3k\frac{\pi}{2}\right) - j\sin(3k\frac{\pi}{2})\right]$$

$$= \frac{1}{4}\left[1 + 2\cos\left(k\frac{\pi}{2}\right)\right]$$

$$\therefore a_0 = \frac{3}{4}, a_1 = \frac{1}{4}, a_2 = -\frac{1}{4}, a_3 = \frac{1}{4}$$



Fourier series:

$$x[n] = \sum_{k=0}^{3} a_k e^{jk\Omega_0 n}$$

$$= \frac{3}{4} + \frac{1}{4}e^{j\Omega_0 n} - \frac{1}{4}e^{j2\Omega_0 n} + \frac{1}{4}e^{j3\Omega_0 n}$$

$$= \frac{3}{4} + \frac{1}{4}\left[\cos\left(n\frac{\pi}{2}\right) + j\sin(n\frac{\pi}{2})\right]$$

$$-\frac{1}{4}\left[\cos\left(2n\frac{\pi}{2}\right) + \mathrm{j}\sin\left(2n\frac{\pi}{2}\right)\right] + \frac{1}{4}\left[\cos\left(3n\frac{\pi}{2}\right) + j\sin(3n\frac{\pi}{2})\right]$$

$$\sin\left(n\frac{\pi}{2}\right) = \sin\left(n\frac{\pi}{2} - 2\pi n\right) = \sin\left(-3n\frac{\pi}{2}\right) = -\sin\left(3n\frac{\pi}{2}\right)$$

$$\cos\left(3n\frac{\pi}{2}\right) = \cos\left(3n\frac{\pi}{2} - 2\pi k\right) = \cos\left(-n\frac{\pi}{2}\right) = \cos\left(n\frac{\pi}{2}\right)$$

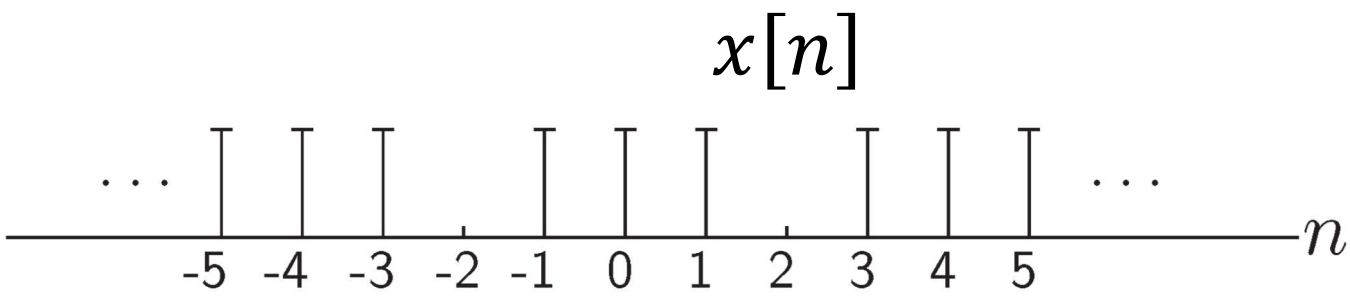
$$\sin\left(2n\frac{\pi}{2}\right) = 0$$

$$\cos\left(2n\frac{\pi}{2}\right) = (-1)^n$$

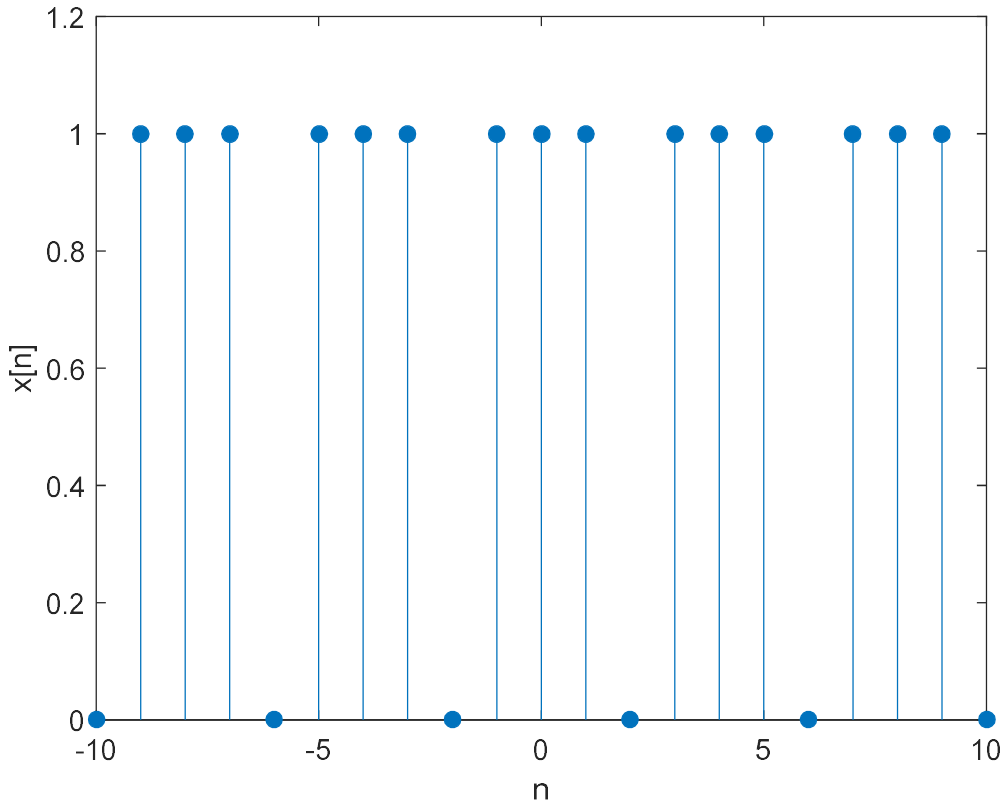
Then,

$$x[n] = \frac{3}{4} + \frac{1}{2}\left[\cos\left(n\frac{\pi}{2}\right)\right] - \frac{1}{4}(-1)^n$$

MATLAB




```
n=-10:10;
x=3/4+1/2*cos(n*pi/2)-1/4*(-1).^n;
stem(n,x,'fill')
xlabel('n')
ylabel('x[n]')
```



Example: Consider the periodic signal

$v[0] = 1$

$v[1] = 1$

$v[2] = 1$

$v[3] = 0$

From the previous example

$x[0] = 1$

$x[1] = 1$

$x[2] = 0$

$x[3] = 1$

and $v[n] = v[n-4]$ for all n

We can see that $v[n] = x[n-1]$, for all n

$$x[n] = \sum_{k=0}^{3} a_k e^{jk\Omega_0 n}$$

Then,

$$v[n] = \sum_{k=0}^{3} a_k e^{jk\Omega_0 (n-1)}$$

$$v[n] = \sum_{k=0}^{3} a_k e^{jk\Omega_0(n-1)}$$

$$v[n] = \sum_{k=0}^{3} a_k e^{-jk\Omega_0} e^{jk\Omega_0 n}$$

$$v[n] = \sum_{k=0}^{3} \tilde{a}_k e^{jk\Omega_0 n}$$

Where $\tilde{a}_k = a_k e^{-jk\Omega_0}$

Then $\tilde{a}_0 = \frac{3}{4}, \tilde{a}_1 = \frac{-j}{4}, \tilde{a}_2 = \frac{1}{4}, \tilde{a}_3 = \frac{j}{4}$

$$v[n] = \frac{3}{4} - \frac{j}{4}\left[\cos\left(n\frac{\pi}{2}\right) + j\sin(n\frac{\pi}{2})\right]$$
$$+\frac{1}{4}\left[\cos\left(2n\frac{\pi}{2}\right) + \mathrm{j}\sin\left(2n\frac{\pi}{2}\right)\right] + \frac{j}{4}\left[\cos\left(3n\frac{\pi}{2}\right) + j\sin(3n\frac{\pi}{2})\right]$$

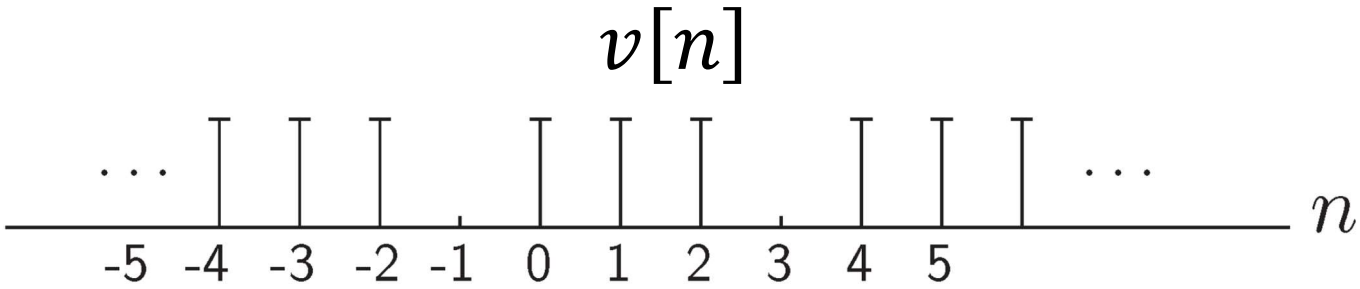
$$\sin\left(3n\frac{\pi}{2}\right) = -\sin\left(n\frac{\pi}{2}\right), \cos\left(3n\frac{\pi}{2}\right) = \cos\left(n\frac{\pi}{2}\right),$$
$$\sin\left(2n\frac{\pi}{2}\right) = 0, \cos\left(2n\frac{\pi}{2}\right) = (-1)^n$$

$$v[n] = \frac{3}{4} - \frac{j}{4}\left[\cos\left(n\frac{\pi}{2}\right) + j\sin(n\frac{\pi}{2})\right]$$
$$+\frac{1}{4}[(-1)^n + 0] + \frac{j}{4}\left[\cos\left(n\frac{\pi}{2}\right) - j\sin(n\frac{\pi}{2})\right]$$

$$v[n] = \frac{3}{4} - \frac{j}{2}\left[\mathrm{j}\sin\left(n\frac{\pi}{2}\right)\right] + \frac{1}{4}(-1)^n$$

$$\boldsymbol{v[n] = \frac{3}{4} + \frac{1}{2}\left[\sin\left(n\frac{\pi}{2}\right)\right] + \frac{1}{4}(-1)^n}$$

MATLAB




```
n=-10:10;
v=3/4+1/2*sin(n*pi/2)+1/4*(-1).^n;
stem(n,v,'fill')
xlabel('n')
ylabel('v[n]')
```

