EE 362: Signals and Systems

Fourier Transform

Lecture 8

Inverse Fourier Transform

- Given a signal $x(t)$ with Fourier transform $\mathrm{X}(\omega)$, $x(t)$ can be recomputed from $\mathrm{X}(\omega)$ by application of the inverse Fourier transform.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

The transform pair notation is,

$$x(t) \leftrightarrow X(\omega)$$

Then

$$P_\tau(t) \leftrightarrow \tau \, \mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

Properties of Fourier Transform

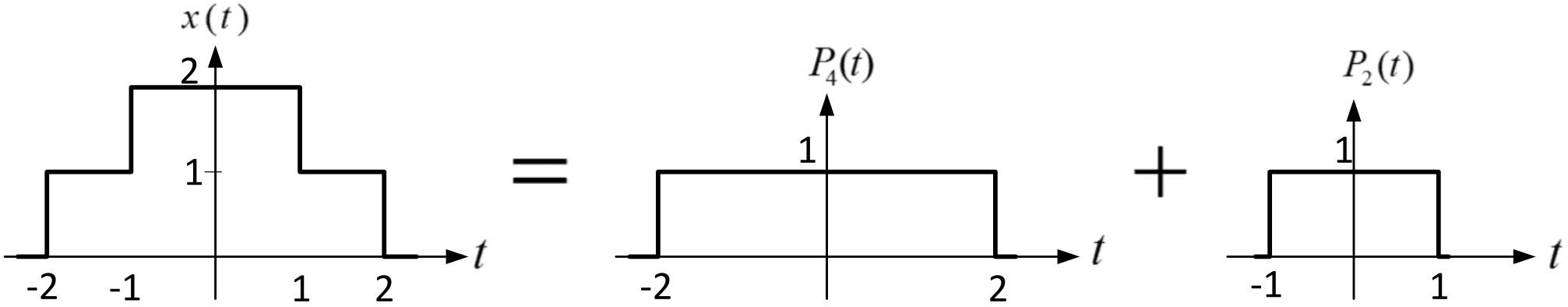
- Linearity
- Left or right shift in time
- Time scaling
- Time reversal
- Multiplication by a power of t
- Multiplication by a complex Exponential
- Multiplication by a sinusoid
- Differentiation in time domain
- Integration in time domain
- Convolution in the time domain
- Multiplication in the time domain
- Parseval's theorem
- Duality

- **Linearity**

$$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$$

Example:

$$x(t) = P_4(t) + P_2(t)$$



$$X(\omega) = 4\,\mathrm{sinc}\left(\frac{4\omega}{2\pi}\right) + 2\,\mathrm{sinc}\left(\frac{2\omega}{2\pi}\right)$$

- **Left or right shift in time**

If $x(t) \leftrightarrow X(\omega)$, then for any positive or negative real number c,

$$x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$$

To verify, apply the definition of Fourier transform with the change of variable $\bar{t} = t - c$

Example: $x(t) = P_2(t-1)$

$$X(\omega) = 2\,\mathrm{sinc}\left(\frac{2\omega}{2\pi}\right)e^{-j\omega}$$

Note that $|e^{-j\omega}| = 1$ for all ω, then $|X(\omega)|$ of $P_2(t-1)$ will be the same for $P_2(t)$.

- **Time scaling**

If $x(t) \leftrightarrow X(\omega)$, for any positive real number a,

$$x(at) \leftrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

To verify, apply the definition of Fourier transform with the change of variable $\bar{t} = at$.

Example: consider $P_2(t)$ and $P_2(2t)$

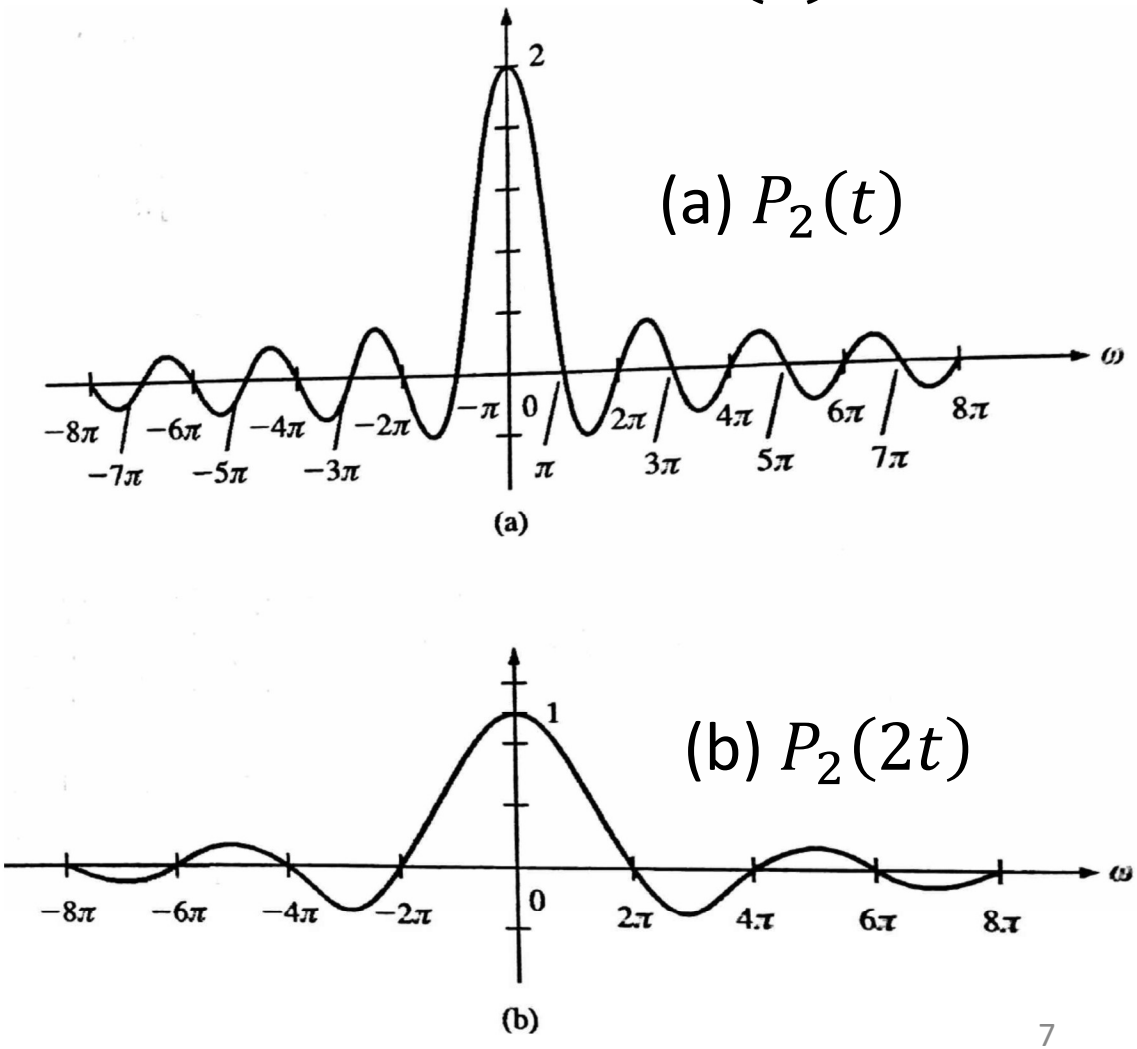
Can you plot $P_2(t)$ and $P_2(2t)$?

$$P_2(t) \leftrightarrow 2\,\mathrm{sinc}\left(\frac{2\omega}{2\pi}\right)$$

$$P_2(2t) \leftrightarrow \frac{2}{2}\mathrm{sinc}\left(\frac{2\omega}{4\pi}\right)$$

- If $a > 1$, $x(at)$ is a time compression of $x(t)$ and Fourier transform of $x(at)$ is a frequency expansion of the Fourier transform of $x(t)$.

Hence, shorter-time-duration signals have wider bandwidths than longer-time-duration signals.



- **Time reversal**

The time reversal of $x(t)$ is $x(-t)$.

$$x(t) \leftrightarrow X(\omega)$$
$$x(-t) \leftrightarrow X(-\omega)$$

To prove it, apply Fourier transform and replace t by $-t$.

If $x(t)$ is real valued, then

$$X(-\omega) = \overline{X(\omega)}$$

Where $\overline{X(\omega)}$ is the complex conjugate of $X(\omega)$, then

$$x(-t) \leftrightarrow \overline{X(\omega)}$$

Example: Given a real number $b > 0$, consider the signal

$$x(t) = \begin{cases} 0, & t > 0 \\ e^{bt}, & t \leq 0 \end{cases}$$

Note that,

$$x(-t) = e^{-bt}u(t)$$

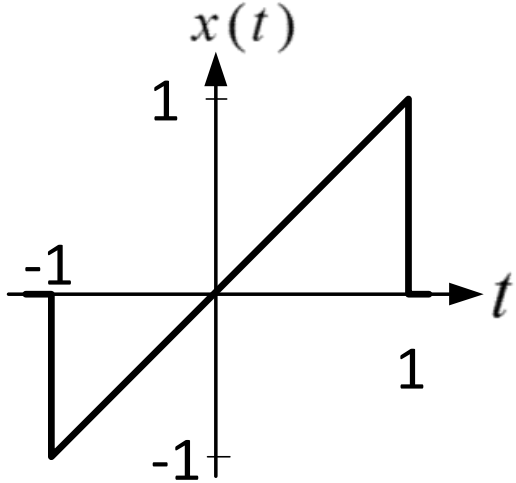
From the previous lecture, the Fourier transform of $e^{-bt}u(t)$ is $1/(b + j\omega)$. Then, the Fourier transform of $x(t)$ is

$$X(\omega) = \frac{1}{b - j\omega}$$

- **Multiplication by a power of t**

If $x(t) \leftrightarrow X(\omega)$, for any positive integer n,

$$t^n x(t) \leftrightarrow (j)^n \frac{d^n}{d\omega^n} X(\omega)$$



Example: let $x(t) = tP_2(t)$

$$X(\omega) = j\frac{d}{d\omega}\left(2\,\mathrm{sinc}\left(\frac{\omega}{\pi}\right)\right)$$

$$= j2\left(\frac{\sin(\omega)}{\omega}\right) = j2\left(\frac{\omega\cos(\omega) - \sin(\omega)}{\omega^2}\right)$$

- **Multiplication by a complex exponential**

If $x(t) \leftrightarrow X(\omega)$, then for any real number ω_0,

$$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

Therefore, multiplication by a complex exponential in time domain corresponds to a frequency shift in the frequency domain.

- **Multiplication by a sinusoid**

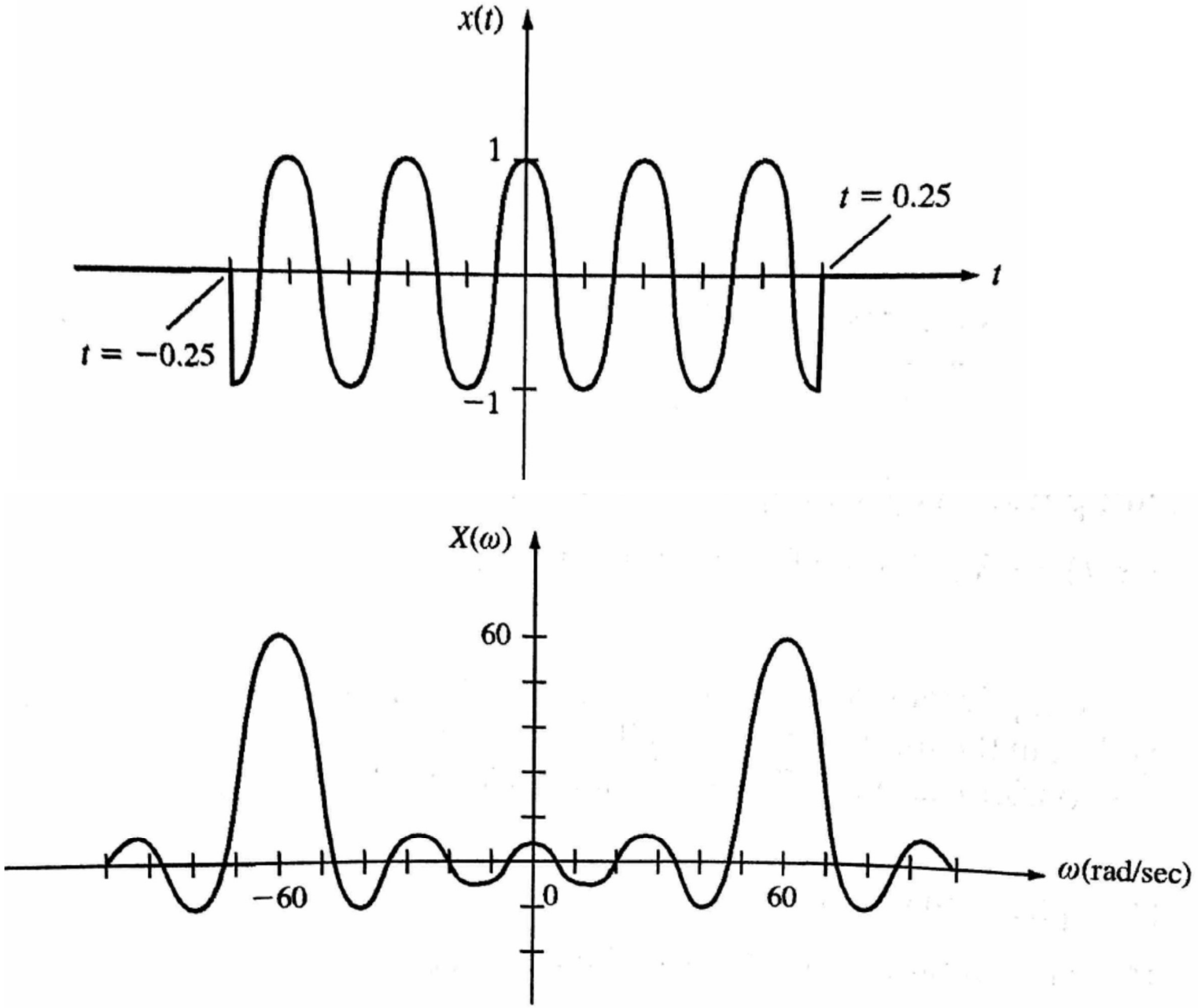
If $x(t) \leftrightarrow X(\omega)$, then for any real number ω_0,

$$x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}[X(\omega+\omega_0) - X(\omega-\omega_0)]$$

$$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega+\omega_0) + X(\omega-\omega_0)]$$

Example: Consider the signal $x(t) = P_\tau(t)\cos(\omega_0 t)$ for $\tau = 0.5$ and $\omega_0 = 60\ rad/sec$, the graph is shown in the next slide and Fourier transform is

$$\frac{1}{2}\left[\tau\ \text{sinc}\left(\frac{\tau(\omega+\omega_0)}{2\pi}\right) + \tau\ \text{sinc}\left(\frac{\tau(\omega-\omega_0)}{2\pi}\right)\right]$$


x(t)
1
t = 0.25
t
t = −0.25
−1
X(ω)
60
−60
0
60
ω(rad/sec)

- **Differentiation in the time domain**

If $x(t) \leftrightarrow X(\omega)$, then for any positive integer n,

$$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega)$$

For the case $n = 1$, it follows from the above expression that differentiation in the time domain corresponds to multiplication by $j\omega$ in the frequency domain. For the case $n = 1$, the Fourier transform of $\frac{dx(t)}{dt}$ is,

$$\int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j\omega t}\, dt$$

Use integration by part. Let $v = e^{-j\omega t}$ and

$w = x(t)$, then $dv = -j\omega e^{-j\omega t}dt$ and $dw = \left[\frac{dx(t)}{dt}\right]dt$

$$\int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j\omega t}\, dt = vw|_{t=-\infty}^{t=\infty} - \int_{-\infty}^{\infty} w dv$$

$$= e^{-j\omega t}x(t)|_{t=-\infty}^{t=\infty} - \int_{-\infty}^{\infty} x(t)(-j\omega)\, e^{-j\omega t} dt$$

If $x(t) \to 0$ at $t \to \pm\infty$, $\int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j\omega t}\, dt = (j\omega)X(\omega)$

The proof for n>1 can be done by repeating integration by part.

- **Integration in the time domain**

Given a signal $x(t)$, the integral of $x(t)$ is the function,

$$\int_{-\infty}^{t} x(\lambda)d\lambda$$

$$\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$$

Where $X(0)$ is the DC component of the signal and $\delta(\omega)$ is the impulse function in the frequency domain.

- **Convolution in the time domain**

Given two signals $x(t)$ and $v(t)$ with Fourier transforms $X(\omega)$ and $V(\omega)$, the Fourier transform of the convolution is,

$$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$$

Then, the convolution in the time domain corresponds to multiplication in the frequency domain.

The above pair of transformation can derived by applying the concept of Fourier transform.

- **Multiplication in the time domain**

If $x(t) \leftrightarrow X(\omega)$ and $v(t) \leftrightarrow V(\omega)$, then

$$x(t)v(t) \leftrightarrow \frac{1}{2\pi}[X(\omega) * V(\omega)]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)V(\omega - \lambda)d\lambda$$

Then, multiplication in the time domain corresponds to convolution in the Fourier transform domain

- **Parseval's Theorem**

Suppose that $x(t) \leftrightarrow X(\omega)$ and $v(t) \leftrightarrow V(\omega)$, then

$$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$$

The Fourier transform of $\int_{-\infty}^{\infty} x(t)v(t)dt$ is

$$\int_{-\infty}^{\infty} x(t)v(t)e^{-j\omega t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)V(\omega-\lambda)d\lambda$$

$$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$\int_{-\infty}^{\infty} x^2(t)dt$ represents the energy of the signal

- **Duality**

Suppose that $x(t) \leftrightarrow X(\omega)$. A new continuous-time signal can be defined by setting $\omega = t$ in $X(\omega)$. This results in the continuous-time signal $X(t)$. The duality property states the following,

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

Example:

$$\tau \operatorname{sinc}\left(\frac{\tau t}{2\pi}\right) \leftrightarrow 2\pi P_\tau(-\omega)$$

But $P_\tau(-\omega) = P_\tau(\omega)$ because it is an even function.

Then sinc function in time domain corresponds to a rectangular pulse function in the frequency domain.

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Right or left shift in time	$x(t - c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X\left(\frac{\omega}{a}\right)\ a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by a power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n}X(\omega)\ n = 1, 2, \ldots$
Multiplication by a complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)\ \ \omega_0$ real
Multiplication by $\sin(\omega_0 t)$	$x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}[X(\omega + \omega_0) - X(\omega - \omega_0)]$
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in the time domain	$\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega)\ n = 1, 2, \ldots$
Integration in the time domain	$\int_{-\infty}^{t} x(\lambda)\,d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in the time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in the time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)\,d\omega$
Special case of Parseval's theorem	$\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\,d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Generalized Fourier Transform

- The unit-step function and $\sin(\omega t)$ and $\cos(\omega t)$ do not have Fourier transformation in the ordinary sense.
- Since these functions arise in the study of signals and systems, it is very desirable to be able to define the Fourier transform of these signals.
- This can be done using the notion of the generalized Fourier transform.

$$\delta(t) = 0, t \neq 0$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(\lambda) d\lambda = 1 \text{ , for all } \varepsilon > 0$$

The Fourier transform of $\delta(t)$ is given by

$$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt$$

Since $\delta(t) = 0$ for all $t \neq 0$,

$$\delta(t) e^{-j\omega t} = \delta(t)$$

and the Fourier transform integral reduces to

$$\int_{-\infty}^{\infty} \delta(t)\, dt$$

- This integral is equal to 1

$$\delta(t) \leftrightarrow 1$$

Now applying the duality property

$$x(t) = 1, -\infty < t < \infty \leftrightarrow \ \ 2\pi\delta\,(\omega)$$

- The frequency function $2\pi\delta(\omega)$ is called the generalized Fourier transform of the constant signal $x(t) = 1, -\infty < t < \infty$

- Now consider $x(t) = \cos(\omega_0 t), -\infty < t < \infty$ and $x(t) = \sin(\omega_0 t)\,, -\infty < t < \infty.$

$$x(t) = 1, -\infty < t < \infty \leftrightarrow \;\; 2\pi\delta\,(\omega)$$

We can get,

$$\cos(\omega_0 t) \leftrightarrow \;\; \pi[\delta\,(\omega + \omega_0) + \delta\,(\omega - \omega_0)]$$

$$\sin(\omega_0 t) \leftrightarrow \;\; j\pi[\delta\,(\omega + \omega_0) - \delta\,(\omega - \omega_0)]$$

TABLE 3.2 Common Fourier Transform Pairs

$1, -\infty < t < \infty \leftrightarrow 2\pi\delta(\omega)$

$-0.5 + u(t) \leftrightarrow \dfrac{1}{j\omega}$

$u(t) \leftrightarrow \pi\delta(\omega) + \dfrac{1}{j\omega}$

$\delta(t) \leftrightarrow 1$

$\delta(t - c) \leftrightarrow e^{-j\omega c}$, c any real number

$e^{-bt}u(t) \leftrightarrow \dfrac{1}{j\omega + b}, b > 0$

$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$, ω_0 any real number

$p_\tau(t) \leftrightarrow \tau \operatorname{sinc}\dfrac{\tau\omega}{2\pi}$

$\tau \operatorname{sinc}\dfrac{\tau t}{2\pi} \leftrightarrow 2\pi p_\tau(\omega)$

$\left(1 - \dfrac{2|t|}{\tau}\right)p_\tau(t) \leftrightarrow \dfrac{\tau}{2}\operatorname{sinc}^2\left(\dfrac{\tau\omega}{4\pi}\right)$

$\dfrac{\tau}{2}\operatorname{sinc}^2\left(\dfrac{\tau t}{4\pi}\right) \leftrightarrow 2\pi\left(1 - \dfrac{2|\omega|}{\tau}\right)p_\tau(\omega)$

$\cos(\omega_0 t) \leftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$

$\cos(\omega_0 t + \theta) \leftrightarrow \pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$

$\sin(\omega_0 t) \leftrightarrow j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$

$\sin(\omega_0 t + \theta) \leftrightarrow j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$

Fourier Transform of a Periodic Signal

From the transform pair

$$x(t) = 1, -\infty < t < \infty \ \leftrightarrow 2\pi\delta(\omega)$$

and the multiplication by a complex exponential property

$$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

we can get the following transfer pair,

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

Thus, the generalized Fourier transform of a periodic signal can be computed as follows.

Let $x(t)$ be periodic for $-\infty < t < \infty$ with period T.

Then, $x(t)$ has the complex exponential **Fourier series** as follows,

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

where $\omega_0 = 2\pi/T$. Then, Fourier transform can be taken by the use of the linearity and the transform pair $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$.

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi C_k \delta(\omega - k\omega_0)$$

Then, Fourier transform of a periodic signal is a train of impulse functions located at $\omega = k\omega_0, k = 0, \pm 1, \pm 2, \ldots$

Fourier Transform of the Unit-Step Function

The generalized Fourier transform of the unit-step function $u(t)$ can be computed by the integration property $\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$

Since $u(t)$ is equal to the integral of $\delta(t)$ and Fourier transform of $\delta(t)$ is $\delta(t) \leftrightarrow 1$, then

$$\int_{-\infty}^{t} \delta(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega}(1) + \pi(1)\delta(\omega)$$

Then,

$$u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$

Modulation Technique

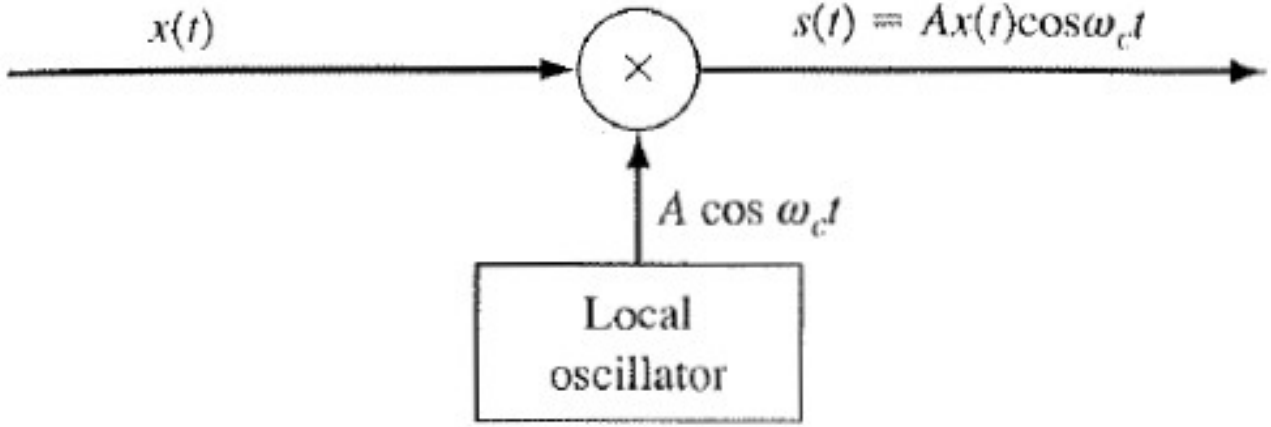
The modulated carrier could be

$$s(t) = Ax(t)\cos(\omega_c t) \qquad \textbf{DSB} - \textbf{SC}$$

Or

$$s(t) = A[1 + kx(t)]\cos(\omega_c t) \qquad \textbf{DSB}$$

What are advantages and disadvantages?



- Analog and Digital Modulation
- Amplitude Modulation (AM)
- Angle Modulation:
  - Frequency Modulation (FM)
  - Phase Modulation (PM)