# EE 362: Signals and Systems

# Fourier Transform

# Lecture 7

# The Fourier Transform

Given a signal $x(t)$, the Fourier transform $X(\omega)$ of $x(t)$ is defined:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad -\infty \le \omega \le \infty$$

Where $\omega$ is called the continuous frequency. $X(\omega)$ may be a complex due to $e^{-j\omega t}$, therefore, we can say:

$$X(\omega) = \left|X(\omega)\right| \angle X(\omega)$$

Where $|X(\omega)|$ is the magnitude function and $\angle X(\omega)$ is the angle function of the Fourier transform $X(\omega)$.

Note: A signal $x(t)$ is said to have a Fourier transform in the ordinary sense if the following integral converges.

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

Example 1: Let $x(t)$ be a DC signal $x(t)=1 \quad -\infty<t<\infty$. Since $x(t)$ is not integrable, the Fourier transform of $x(t)$ does not exist.

Example 2: Let $x(t)$ be an exponential signal, $u(t)$ is the unit-step function, and $b$ is a real constant. Find the Fourier transform of $x(t)$.

$$x(t) = e^{-bt}u(t)$$

The Fourier transform

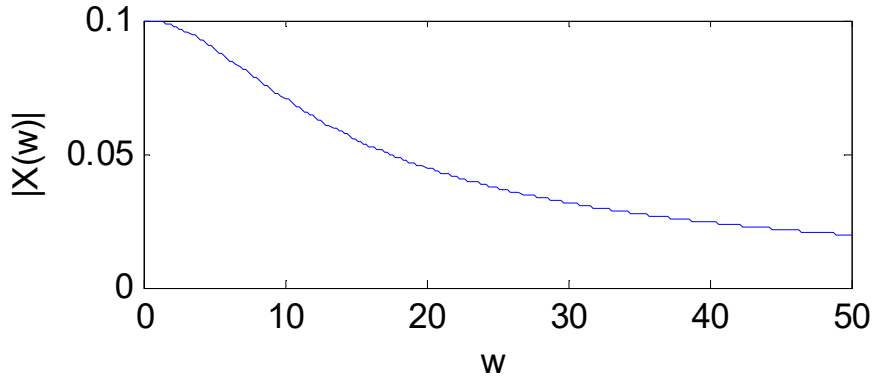
$$X(\omega) = \int_{-\infty}^{\infty} e^{-bt}u(t)\, e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(b+j\omega)t}dt = \frac{1}{b+j\omega}$$

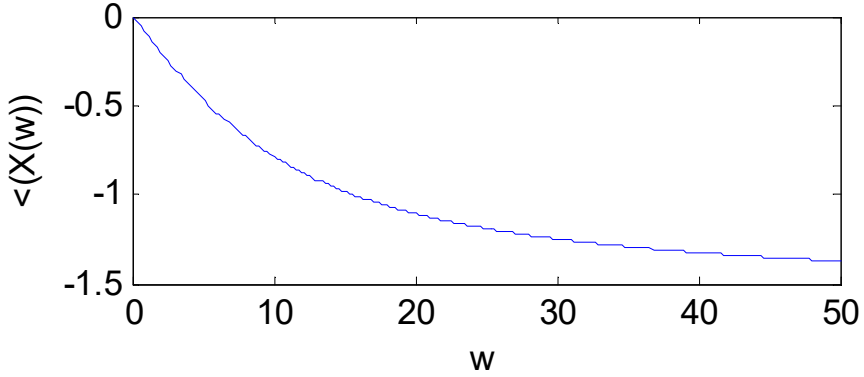
The magnitude spectrum: $|X(\omega)| = \left|\frac{1}{b + j\omega}\right| = \frac{1}{\sqrt{(b^2 + \omega^2)}}$

The angle spectrum: $\angle X(\omega) = -\tan^{-1}(\frac{\omega}{b})$

**MATLAB**

```
w = 0:0.2:50;
b = 10;
X = 1./(b+j*w) ;
subplot(211), plot(w, abs (X));
xlabel('w');ylabel('|X(w)|');
subplot(212), plot(w, angle
(X));xlabel('w');ylabel('<(X(w))');
```



**Practice**: Let $x(t)$ be the rectangular pulse defined by:

$$x(t) = P_\tau(t) \quad \text{wher} \qquad \tau = 2$$

Find the Fourier transform $X(\omega)$

# Rectangular form of the Fourier transform X(ω):

Euler's formula: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \xleftrightarrow{\text{using Euler's formula}}$$

$$\underbrace{\int_{-\infty}^{\infty} x(t)\cos(\omega t)dt}_{R(\omega)} - \underbrace{j\int_{-\infty}^{\infty} x(t)\sin(\omega t)dt}_{jI(\omega)}$$

$$X(\omega) = R(\omega) + jI(\omega)$$

$$R(\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt$$ Real part of $X(\omega)$

$$I(\omega) = -\int_{-\infty}^{\infty} x(t)\sin(\omega t)dt$$ Imaginary part of $X(\omega)$

Even or odd symmetry:

$$R(\omega) = \begin{cases} 2\int_0^{\infty} x(t)\cos(\omega t)dt & \text{when } x(t) \text{ is even} \\ 0 \quad \text{when x(t) is odd} \end{cases}$$

$$I(\omega) = \begin{cases} 0 \quad \text{when } x(t) \text{ is even} \\ 2\int_0^{\infty} x(t)\sin(\omega t)dt & \text{when } x(t) \text{ is odd} \end{cases}$$

Example 3: Let $x(t)$ be the rectangular pulse defined by:

$$P_\tau(t) = \begin{cases} 1 & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$$

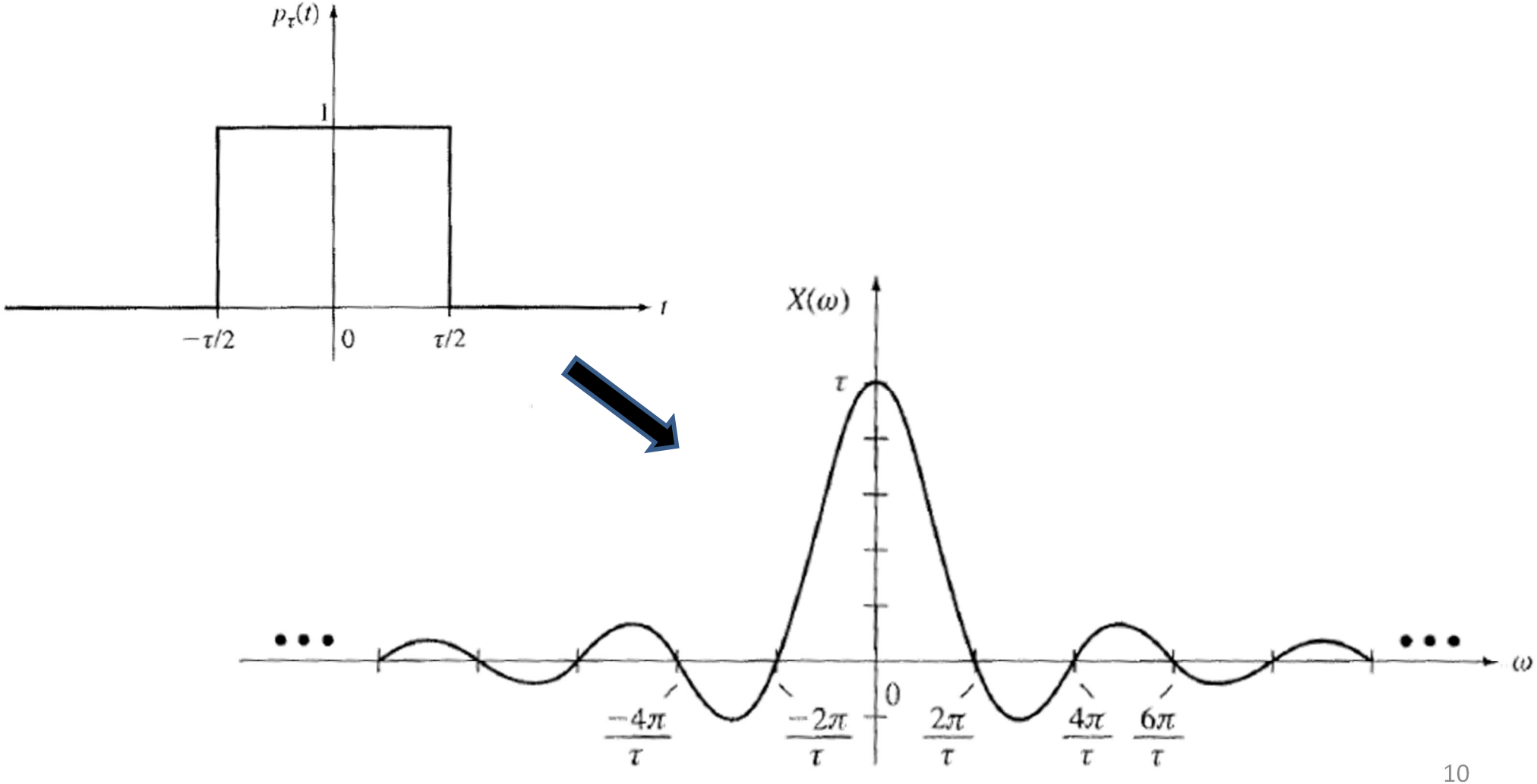
Find the Fourier transform $X(\omega)$

$$R(\omega) = \int_{-\infty}^{\infty} P_\tau(t)\cos(\omega t)dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} 1\cos(\omega t)dt = \frac{2}{\omega}\sin(\frac{\omega\tau}{2})$$

$$I(\omega) = 0$$

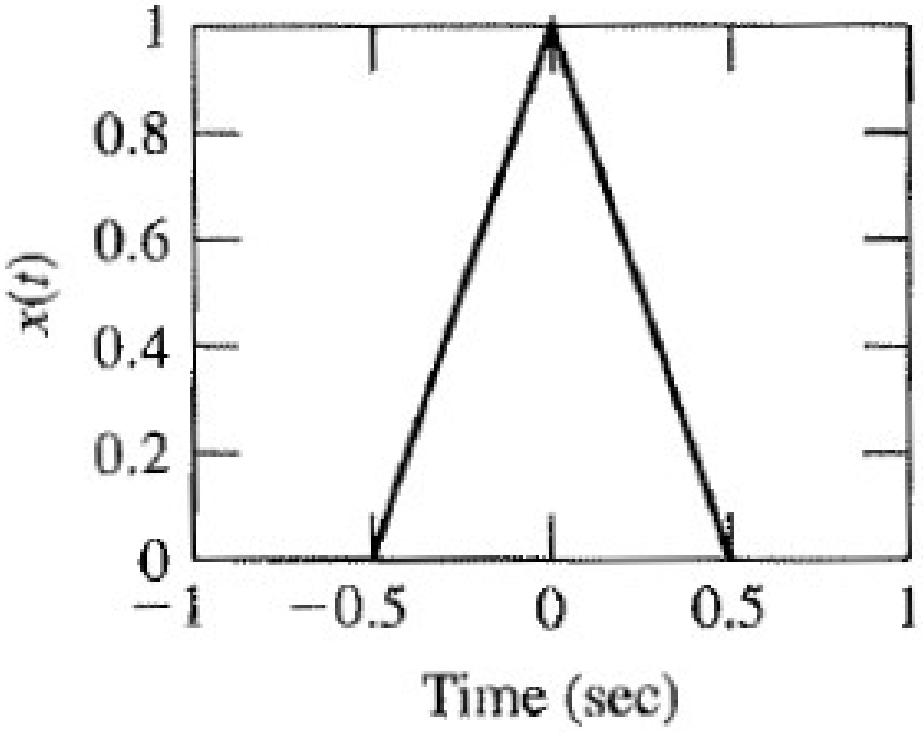
$$X(\omega) = R(\omega) + jI(\omega) = \frac{2}{\omega}\sin(\frac{\omega\tau}{2}) = \frac{2\tau}{\omega\tau}\sin(\frac{\omega\tau}{2}) = \tau\,\mathrm{sinc}(\frac{\omega\tau}{2\pi})$$

This function is called the "Sinc" function



Example 4: Let $x(t)$ be a triangular pulse defined by:

$$x(t) = \begin{cases} 1 - \dfrac{2|t|}{\tau} & -\dfrac{\tau}{2} \le t \le \dfrac{\tau}{2} \\ 0 \quad \text{otherwise} \end{cases}$$




Find the Fourier transform $X(\omega)$, $\tau = 1$

$$R(\omega) = \int_{-\infty}^{\infty} X(t)\ \cos(\omega t)dt = 2\int_{0}^{\frac{1}{2}} (1-2t)\cos(\omega t)dt$$

$$I(\omega) = 0$$

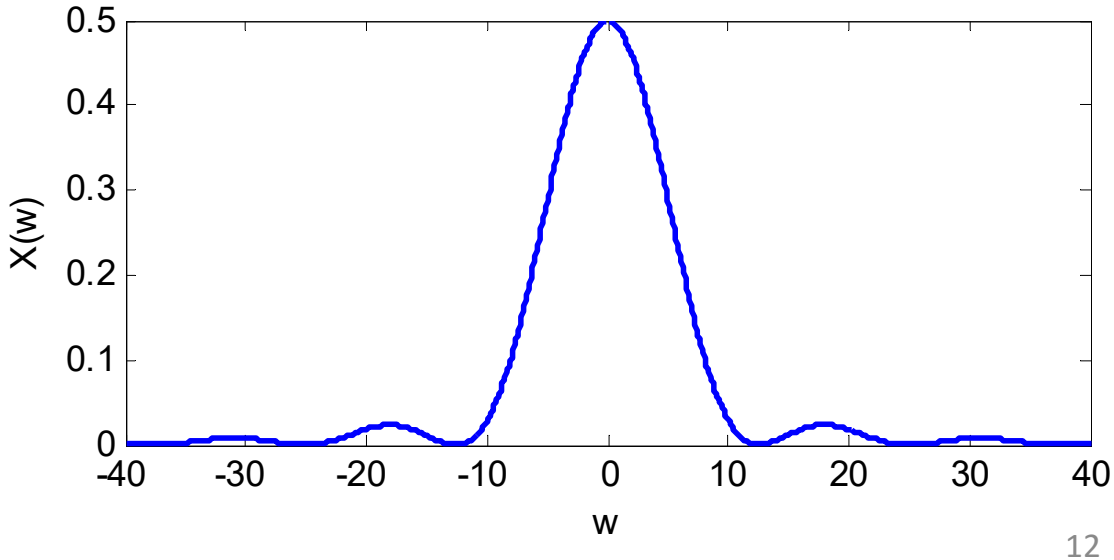
# MATLAB code to calculate the $R(\omega)$

```
syms  t w X
tau = 1;
R= int((1-2*abs(t)/tau)*cos(w*t) ,t,-tau/2,tau/2);
R = simplify(R)
```

```
>> R = -4*(cos(1/2*w)-1)/w^2
```

$$X(\omega) = R(\omega) + jI(\omega) = -4\frac{\cos(0.5\omega) - 1}{\omega^2}$$

```
w=-40:0.1:40;
X=-4*(cos(1/2*w)-1)./w.^2;
 plot(w,X,'linewidth',2)
 xlabel('w')
 ylabel('X(w)')
```



Example 5: Find the Fourier transform $X(\omega)$ of $x(t)$ defined by: $x(t) = e^{-2t}\sin(t)u(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-\omega t}dt = \int_{-\infty}^{\infty} e^{-2t}\sin(t)u(t)e^{-\omega t}dt = \int_{0}^{\infty} e^{-(2+\omega)t}\sin(t)dt$$

**MATLAB**

```
syms x t X omega
x = exp(-2*t)*sin(t);
X = int(x*exp(-j*omega*t),t,0,inf);
X = simplify(X)
```

```
>>X = -1/(-5+omega^2-4*i*omega)
```

$$X(\omega) = \frac{-1}{-5 + \omega^2 - 4j\omega}$$

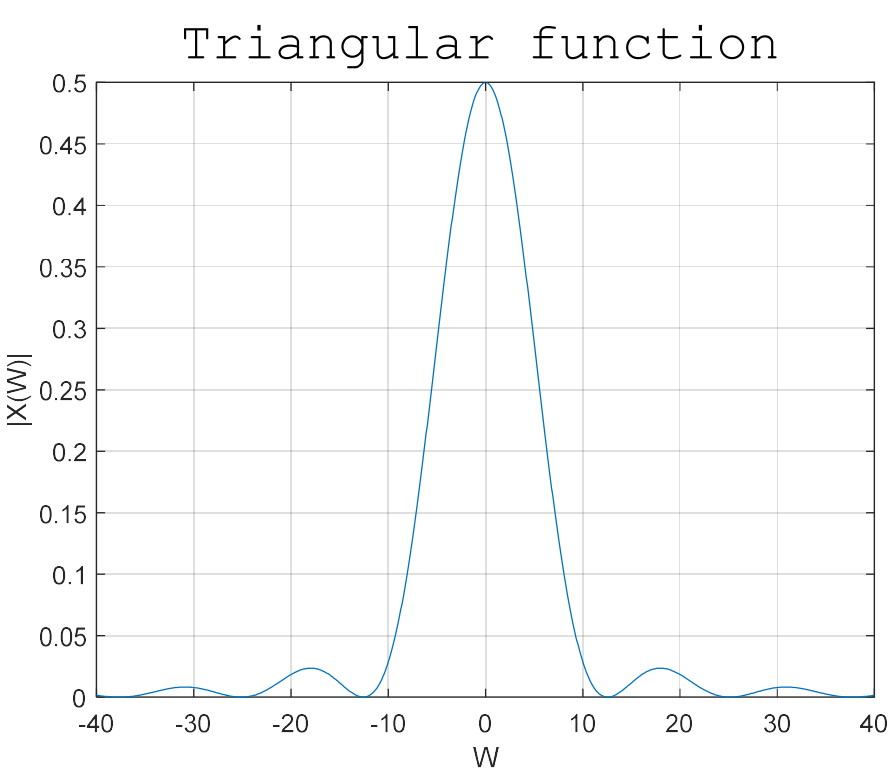
# Spectral Content of Common Signals

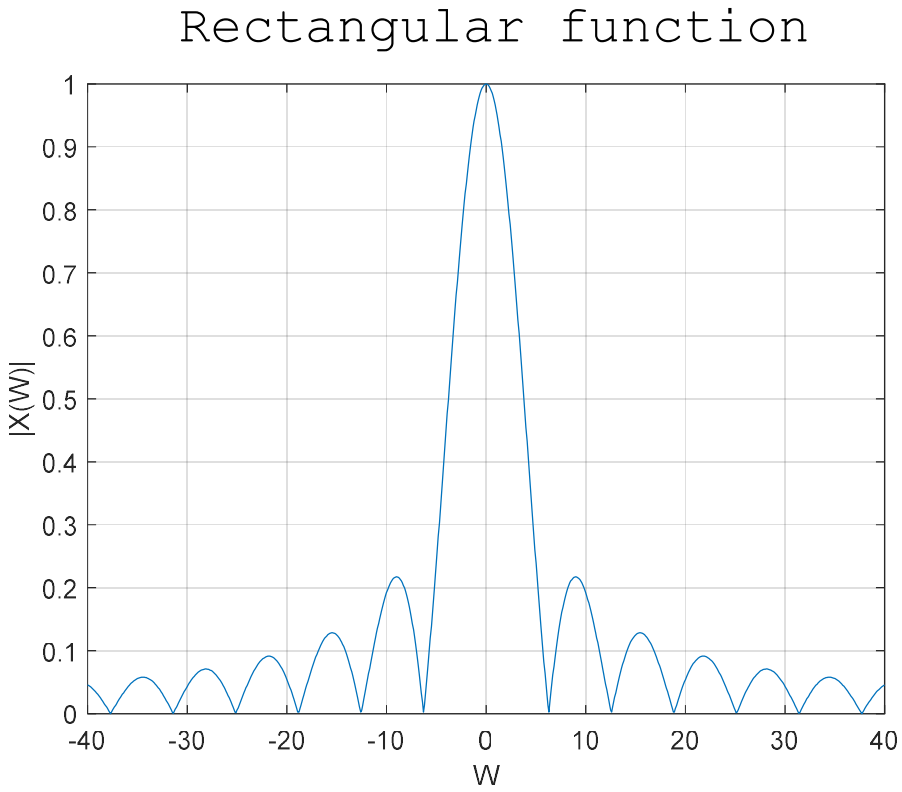
- Triangular vs rectangular

$$x(t) = \begin{cases} 1 - \frac{2|t|}{\tau}, & -\frac{\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$P_\tau(t) = \begin{cases} 1, & -\frac{\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0, & \text{otherwise} \end{cases}$$

```
% Triangular function
w=-40:0.1:40;
F=zeros(1,length(w));
for i=1:length(w)
    F(i)=-4*(cos(1/2*w(i))-1)./w(i)^2;
end
plot(w,abs(F))
```



# Band Limited Signals

- A signal $x(t)$ is said to be bandlimited if its Fourier transform $X(\omega)$ is zero for all $\omega > B$, where $B$ is some positive number, called the bandwidth of the signal.
- If a signal $x(t)$ is bandlimited with bandwidth $B$, the signal does not contain any spectral components with frequency higher than $B$, which satisfies the use of the term bandlimited.
- Any bandlimited signal must be of infinite duration in time – bandlimited signals cannot be time limited.

- If a signal $x(t)$ is not bandlimited, it is said to have infinite bandwidth or an infinite spectrum.
- Since bandlimited signals cannot be time limited, time-limited signals cannot be bandlimited, and thus all time-limited signals have infinite bandwidth.
- Since all (physical) signals are time limited, any such signal must have an infinite bandwidth.
- However, for any well-behaved time-limited signal $x(t)$, it can be proved that the Fourier transform $\mathrm{X}(\omega)$ converges to zero as $\omega \to \infty$.

- Therefore, for any time-limited signal arising in practice, it is always possible to assume that $|X(\omega)| \approx 0$ for all $\omega > B$, where $B$ is chosen to be suitably large.
- Example:

Consider the rectangular pulse function $x(t) = P_\tau(t)$. From previous lectures it was shown that the Fourier transform is

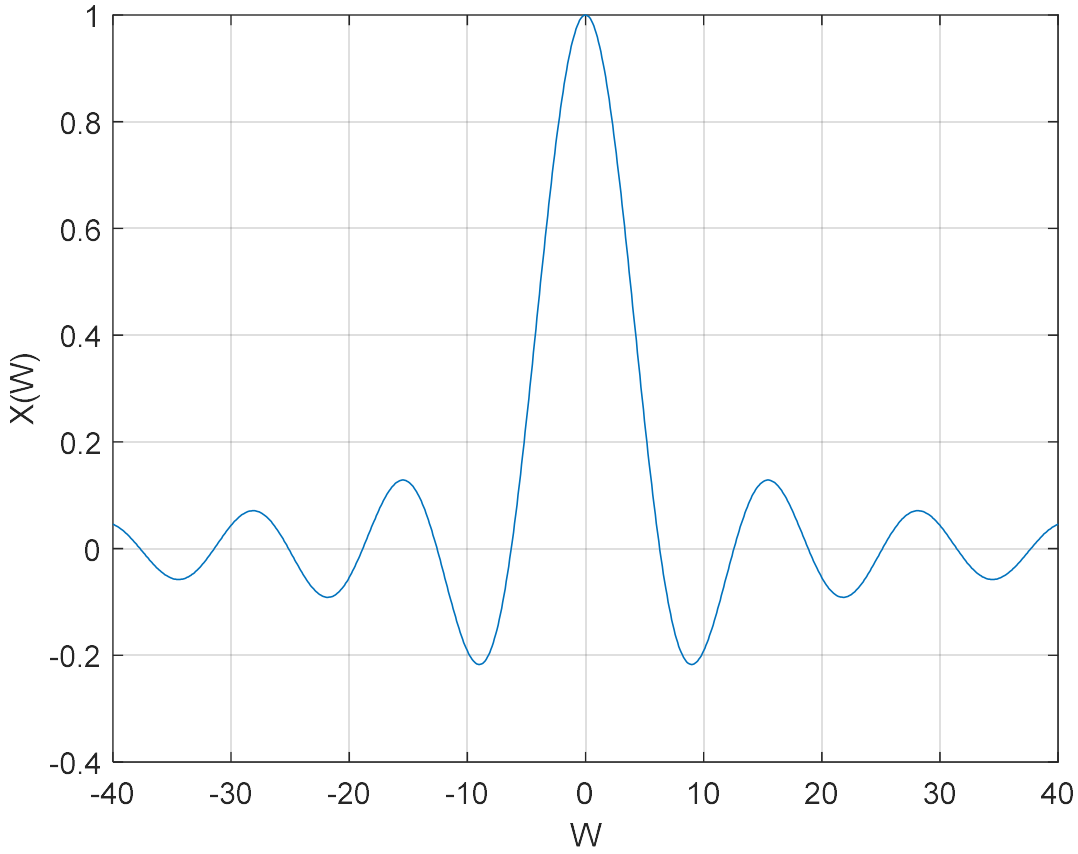
$$X(\omega) = \tau \operatorname{sinc}\left(\frac{\tau\omega}{2\pi}\right)$$

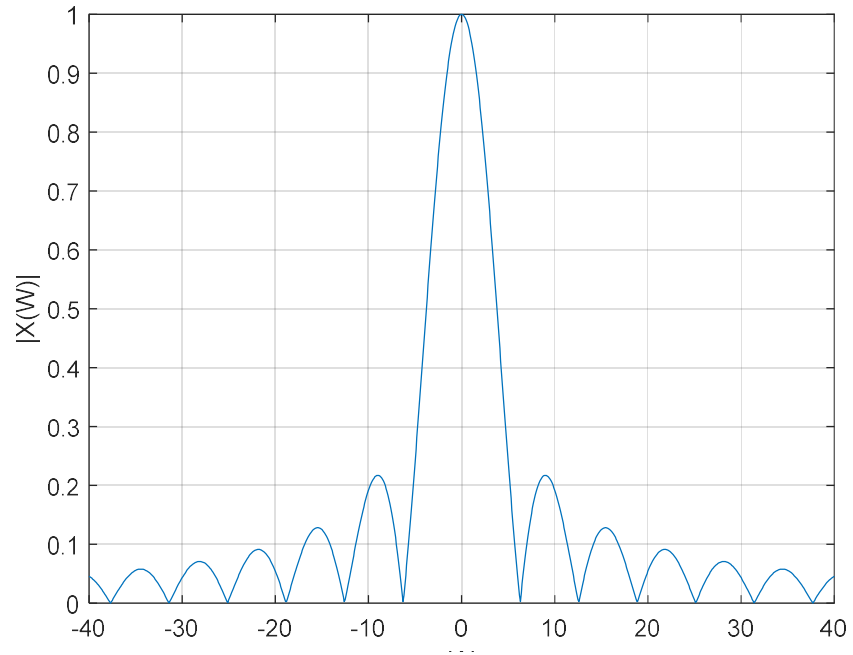
The plots of the amplitude and phase spectra are given in the next slide.

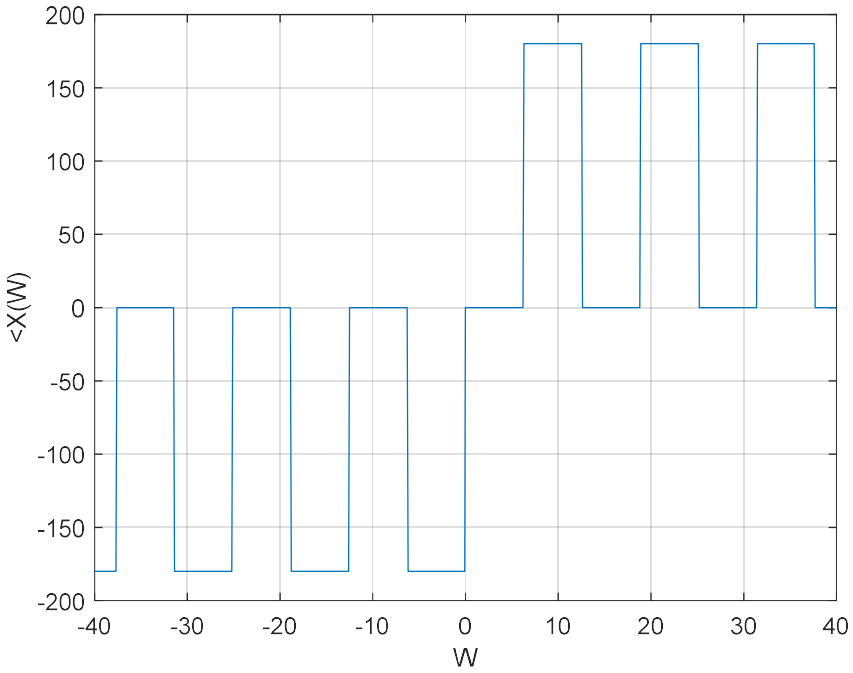
```
tau = 1;
W=-40:0.1:40;
X=tau*sinc(tau*W/(2*pi));
plot(W,X)
xlabel('W')
ylabel('X(W)')
grid

Xmag=abs(X);
figure
plot(W,Xmag)
xlabel('W')
ylabel('|X(W)|')
grid

Xang=zeros(1,length(W));
for i=1:length(W)
    if W(i)<0
        Xang(i)=angle(X(i))-pi;
    else
        Xang(i)=angle(X(i));
    end
end
figure
plot(W,Xang*180/pi)
xlabel('W')
ylabel('<X(W)')
grid
```






# Try different value for $\tau$

- If the time duration $\tau$ of the rectangular pulse is made smaller, the amplitude spectrum spreads out.
- Then, shorter-time-duration signals have more spectral content at higher frequencies than longer-time-duration signals.
- As the frequency $\omega$ increases, the magnitude of sidelobes decreases.
- Then, $|X(\omega)| < c$ for all $\omega > B$, where $c > 0$.
- Therefore, if $B$ is chosen to be sufficiently large, the rectangular pulse can be viewed as being "approximately bandlimited" with bandwidth $B$.

# Spectral Content of Common Signals

- Triangular vs rectangular

$$x(t) = \begin{cases} 1 - \dfrac{2|t|}{\tau}, & -\dfrac{\tau}{2} \le t \le \dfrac{\tau}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$P_\tau(t) = \begin{cases} 1, & -\dfrac{\tau}{2} \le t \le \dfrac{\tau}{2} \\ 0, & \text{otherwise} \end{cases}$$

```
% Triangular function
w=-40:0.1:40;
F=zeros(1,length(w));
for i=1:length(w)
    F(i)=-4*(cos(1/2*w(i))-1)./w(i)^2;
end
plot(w,abs(F))
```

