EE 362: Signals and Systems

The Fourier Series

Lecture 6

The Fourier Series

The representation of signals in terms of frequency components: we can generate the frequency content of a signal by decomposing the signal into frequency components given by sinusoids.

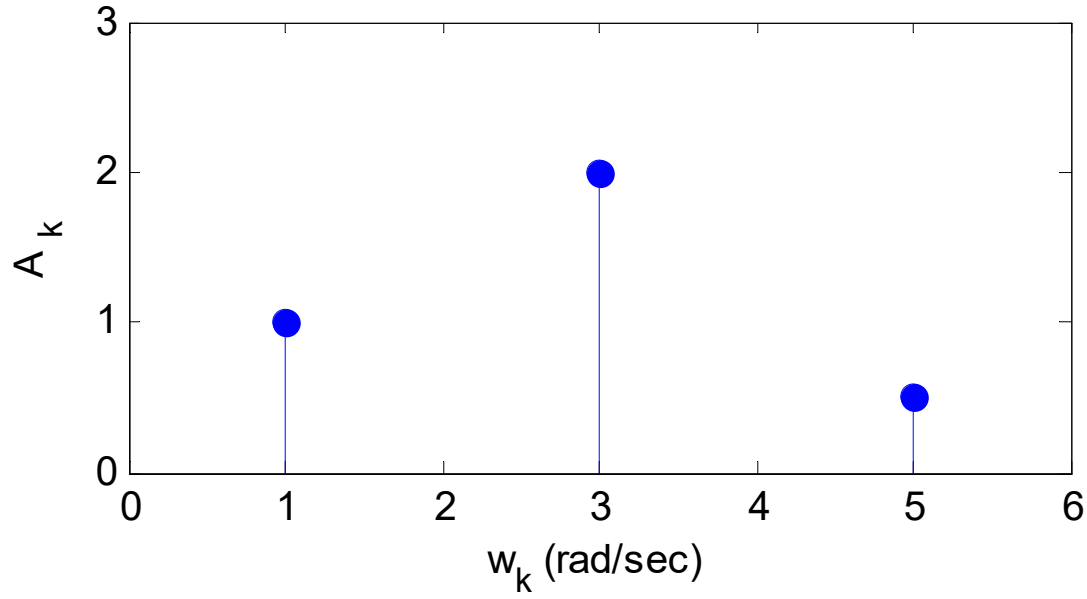
$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_k t + \theta_k)$$

We could say that $x(t)$ has all frequency components with frequencies: ω_1, ω_2,…, ω_N and magnitudes A_1,A_2,… A_N and phases θ_1, θ_2, …, θ_N

Example: $x(t)$ has three frequency components with frequencies ω_1 =1, ω_3 =3 and ω_5 =5 rad/sec and magnitudes A_1=1, A_3=2 and A_5=0.5

$x(t)=\cos(t)+ 2\cos(3t)+ 0.5\cos(5t)$

Note: we can plot A_k versus ω_k, it is called: Amplitude Spectrum



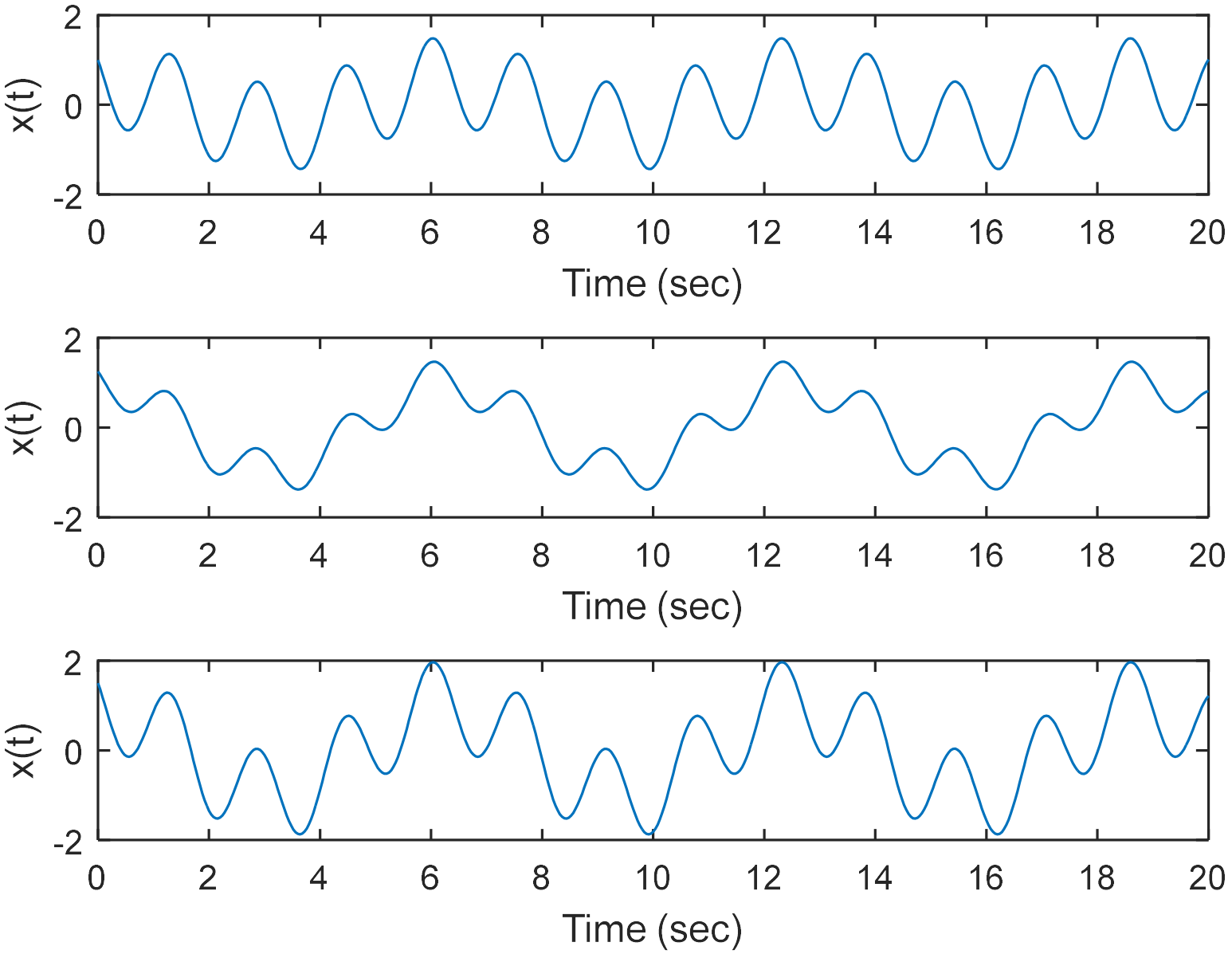
- Example: Consider the following continuous-time signal

$$x(t) = A_1 \cos t + A_2 \cos(4t + \pi/3) + A_3 \cos(8t + \pi/2), -\infty < t < \infty$$

```
t=0:20/500:20;
w1=1; w2=4; w3=8;
A1=0.5; A2=1; A3=0;
x=A1*cos(w1*t)+A2*cos(w2*t+pi/3)+A3*cos(w3*t+pi/2);
subplot(3,1,1); plot(t,x), xlabel('Time (sec)'), ylabel('x(t)')

A1=1; A2=0.5; A3=0;
x=A1*cos(w1*t)+A2*cos(w2*t+pi/3)+A3*cos(w3*t+pi/2);
subplot(3,1,2); plot(t,x), xlabel('Time (sec)'), ylabel('x(t)')

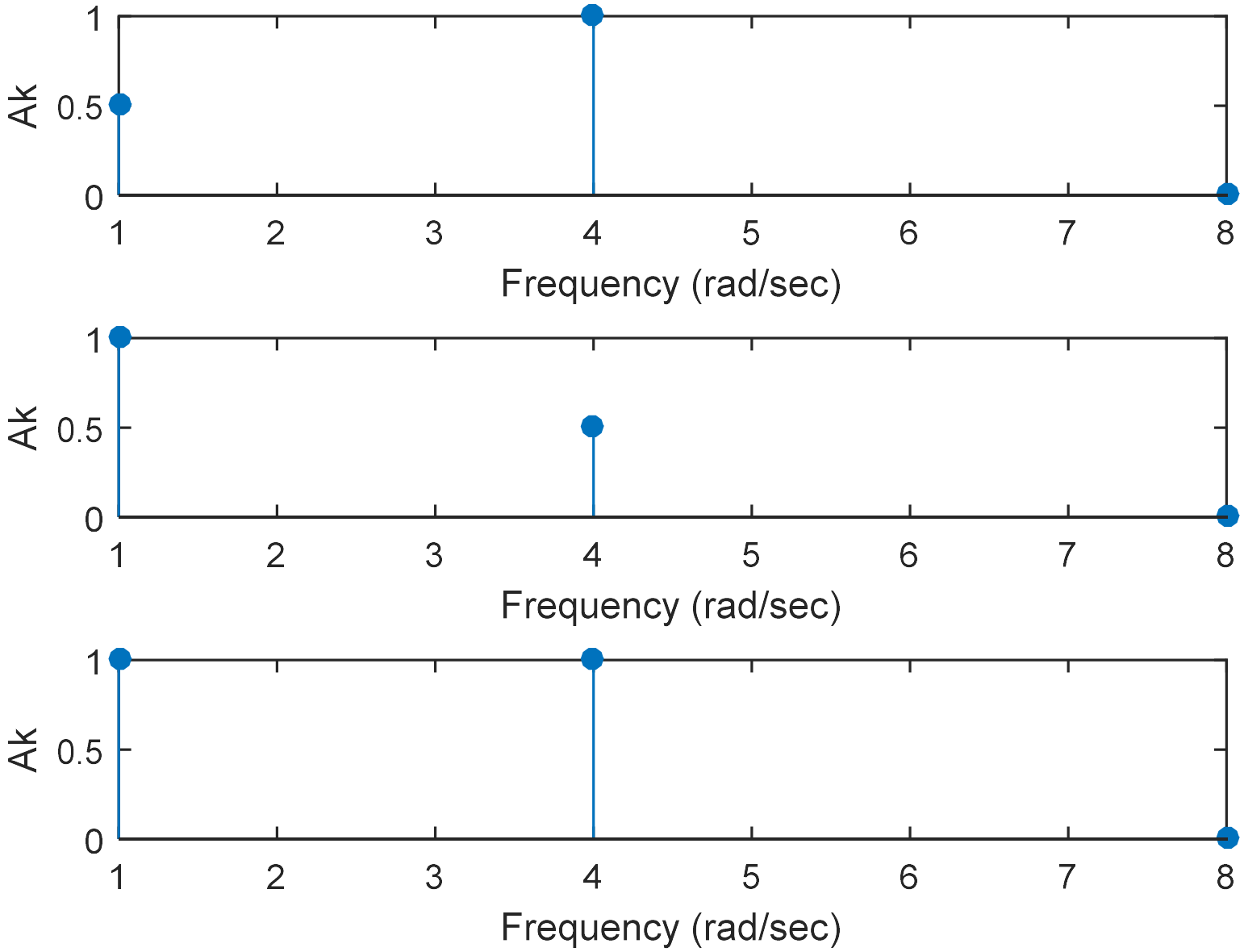
A1=1; A2=1; A3=0;
x=A1*cos(w1*t)+A2*cos(w2*t+pi/3)+A3*cos(w3*t+pi/2);
subplot(3,1,3); plot(t,x), xlabel('Time (sec)'), ylabel('x(t)')
```


2
0
-2
x(t)
0 2 4 6 8 10 12 14 16 18 20
Time (sec)
2
0
-2
x(t)
0 2 4 6 8 10 12 14 16 18 20
Time (sec)
2
0
-2
x(t)
0 2 4 6 8 10 12 14 16 18 20
Time (sec)

```
w1=1; w2=4; w3=8;
A1=0.5; A2=1; A3=0;
subplot(3,1,1); stem([w1 w2 w3],[A1 A2 A3],'filled')
xlabel('Frequency (rad/sec)'), ylabel('Ak')

A1=1; A2=0.5; A3=0;
subplot(3,1,2); stem([w1 w2 w3],[A1 A2 A3],'filled')
xlabel('Frequency (rad/sec)'), ylabel('Ak')

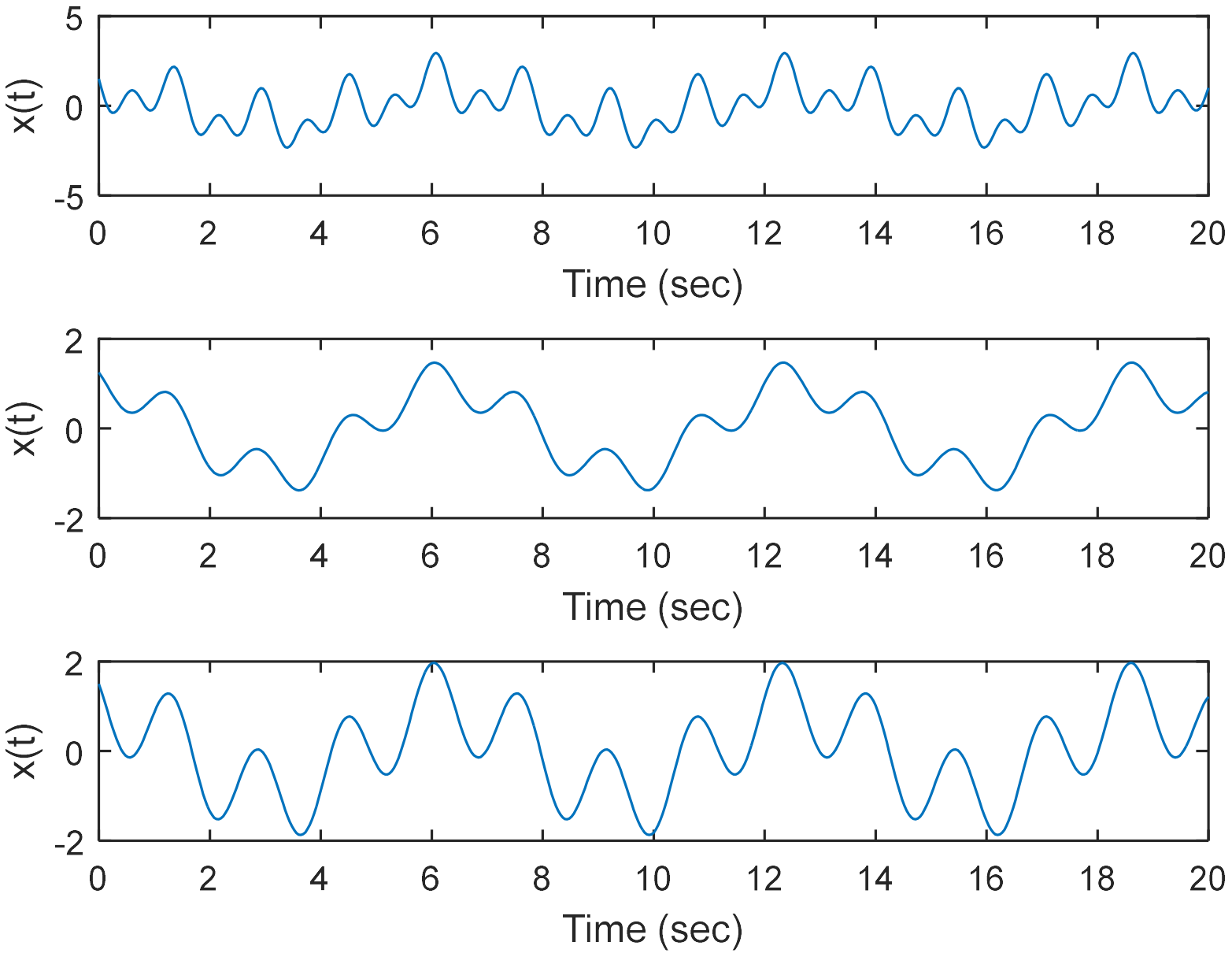
A1=1; A2=1; A3=0;
subplot(3,1,3); stem([w1 w2 w3],[A1 A2 A3],'filled')
xlabel('Frequency (rad/sec)'), ylabel('Ak')
```


Ak
1
0.5
0
1
2
3
4
5
6
7
8
Frequency (rad/sec)
Ak
1
0.5
0
1
2
3
4
5
6
7
8
Frequency (rad/sec)
Ak
1
0.5
0
1
2
3
4
5
6
7
8
Frequency (rad/sec)

```
w1=1; w2=4; w3=8;
A1=0.5; A2=1; A3=0.5;
x=A1*cos(w1*t)+A2*cos(w2*t+pi/3)+A3*cos(w3*t+pi/2);
subplot(3,1,1); plot(t,x), xlabel('Time (sec)'),
ylabel('x(t)')

A1=1; A2=0.5; A3=0.5;
x=A1*cos(w1*t)+A2*cos(w2*t+pi/3)+A3*cos(w3*t+pi/2);
subplot(3,1,1); plot(t,x), xlabel('Time (sec)'),
ylabel('x(t)')

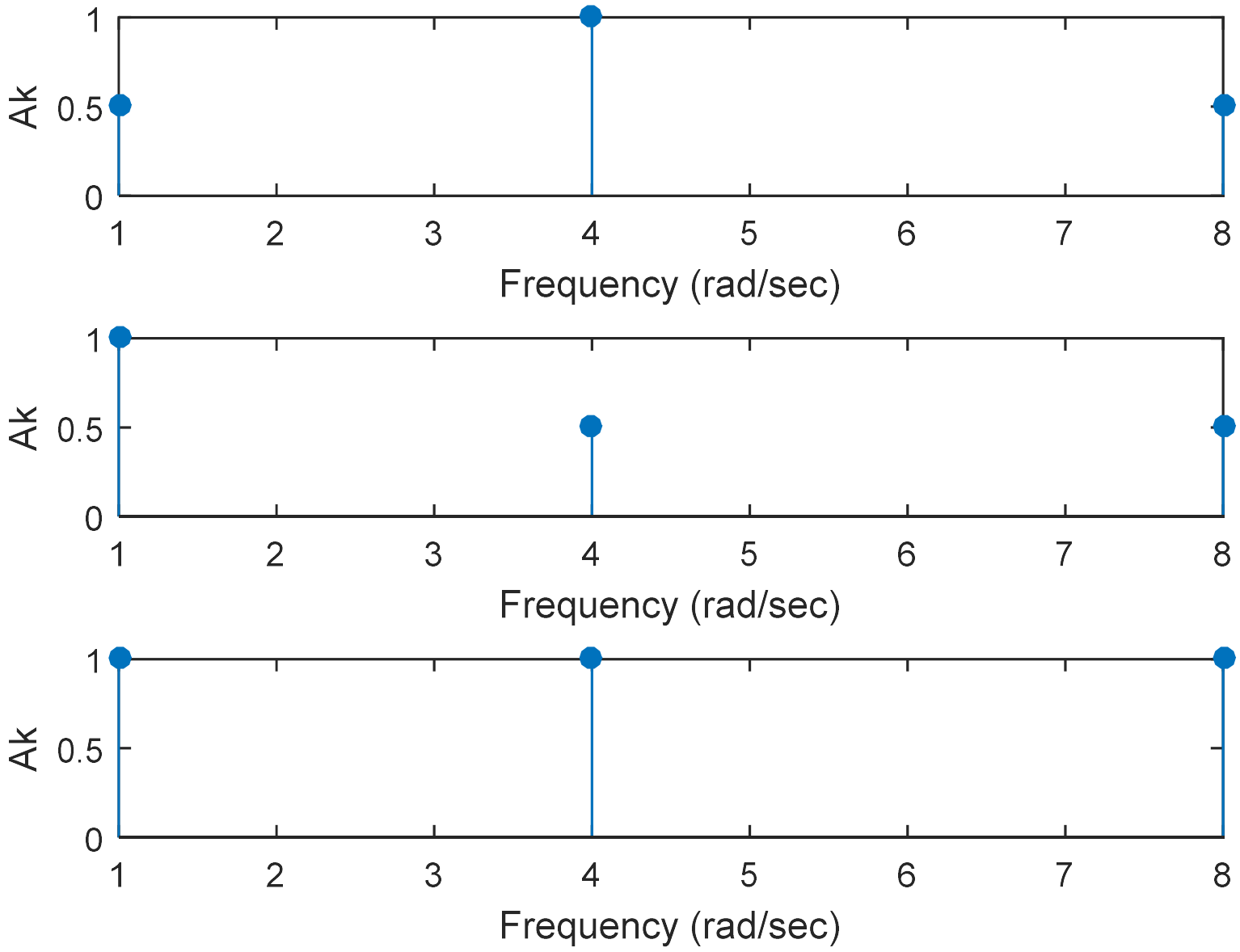
A1=1; A2=1; A3=1;
x=A1*cos(w1*t)+A2*cos(w2*t+pi/3)+A3*cos(w3*t+pi/2);
subplot(3,1,1); plot(t,x), xlabel('Time (sec)'),
ylabel('x(t)')
```


5
0
-5
x(t)
0 2 4 6 8 10 12 14 16 18 20
Time (sec)
2
0
-2
x(t)
0 2 4 6 8 10 12 14 16 18 20
Time (sec)
2
0
-2
x(t)
0 2 4 6 8 10 12 14 16 18 20
Time (sec)

```
w1=1; w2=4; w3=8;
A1=0.5; A2=1; A3=0.5;
subplot(3,1,1); stem([w1 w2 w3],[A1 A2 A3],'filled')
xlabel('Frequency (rad/sec)'), ylabel('Ak')

A1=1; A2=0.5; A3=0.5;
subplot(3,1,2); stem([w1 w2 w3],[A1 A2 A3],'filled')
xlabel('Frequency (rad/sec)'), ylabel('Ak')

A1=1; A2=1; A3=1;
subplot(3,1,3); stem([w1 w2 w3],[A1 A2 A3],'filled')
xlabel('Frequency (rad/sec)'), ylabel('Ak')
```


Ak
1
0.5
0
1
2
3
4
5
6
7
8
Frequency (rad/sec)
Ak
1
0.5
0
1
2
3
4
5
6
7
8
Frequency (rad/sec)
Ak
1
0.5
0
1
2
3
4
5
6
7
8
Frequency (rad/sec)

Fourier Series representation:

Recall that a continuous-time signal $x(t)$ is said to be periodic with period T if:

$$x(t+T) = x(t) \quad \text{for all } t, \; -\infty < t < \infty$$

where T is the fundamental period (smallest number) of $x(t)$

If $x(t)$ is a periodic signal with fundamental period T, then $x(t)$ can be expressed as a sum of sinusoids:

$$x(t) = a_0 + \sum_{k=1}^{N} [a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)] \qquad -\infty < t < \infty$$

$$a_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt$$

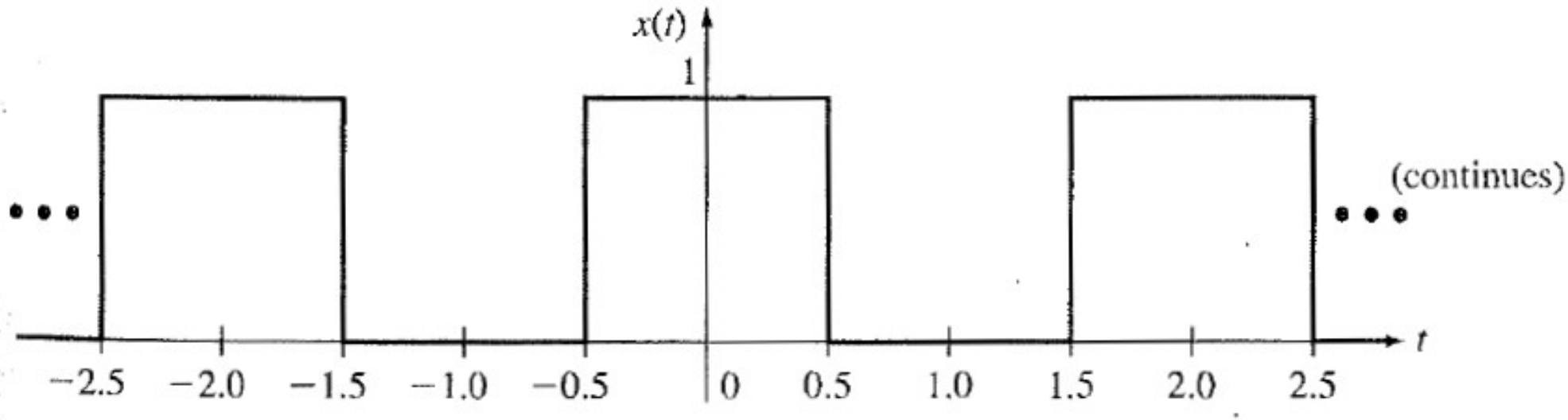
$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)dt = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos(k\omega_0 t)dt \qquad k = 1,2,\ldots$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)dt = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin(k\omega_0 t)dt \qquad k = 1,2,\ldots$$

Note: a_0 is the DC component, T is the fundamental period of $x(t)$, and

$$w_0 = \frac{2\pi}{T}$$

Example: find the Fourier series representation of a rectangular pulse train $x(t)$



$x(t)$ is periodic with $T = 2$, therefore,

$$w_0 = \frac{2\pi}{2} = \pi$$

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{1}{2}\int_{-1}^{1} x(t)dt = \frac{1}{2}\int_{-0.5}^{0.5} 1dt = \frac{1}{2}$$

$$a_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos(k\pi t)dt = \frac{2}{2}\int_{-0.5}^{0.5} (1)\cos(k\pi t)dt = \frac{1}{k\pi}\sin(k\pi t)\,|_{-0.5}^{+0.5} = \frac{2}{k\pi}\sin(\frac{k\pi}{2})$$

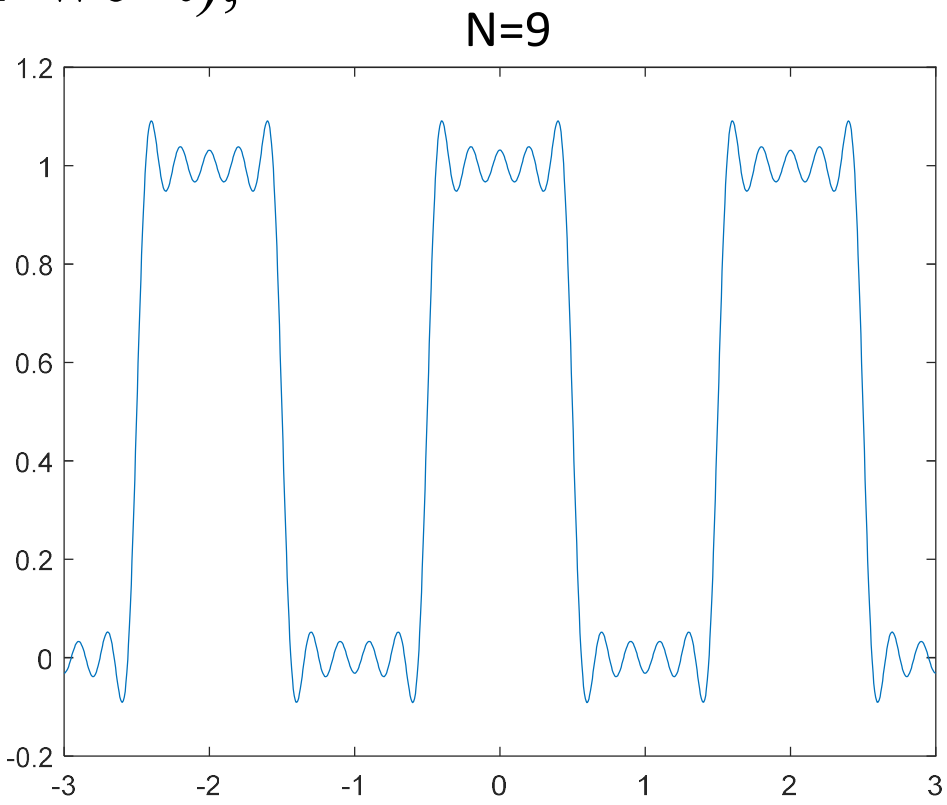
$$b_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin(k\pi t)dt = \frac{2}{2}\int_{-0.5}^{0.5} (1)\sin(k\pi t)dt = \frac{-1}{k\pi}\cos(k\pi t)\,|_{-0.5}^{+0.5} = 0$$

Then,

$$x(t) = \frac{1}{2} + \sum_{k=1}^{N}[\frac{2}{k\pi}\sin(\frac{k\pi}{2})\cos(k\pi t)]$$

```
t=-3:6/1000:3;
N=input('\n Number of harmonics = ');
w0=pi;
f=1/2;
for k=1:N
    f=f+2/k/pi*sin(k*pi/2)*cos(k*w0*t);
end
plot(t,f)
```

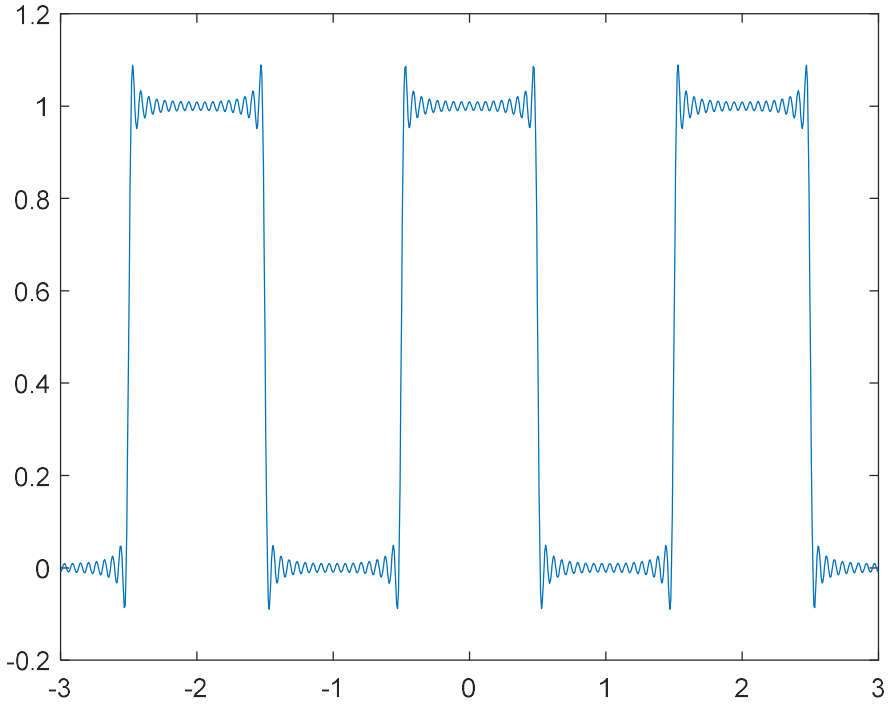
$$x(t) = \frac{1}{2} + \sum_{k=1}^{N} [\frac{2}{k\pi} \sin(\frac{k\pi}{2}) \cos(k\pi t)]$$



Gibbs Phenomenon

By Fourier theorem, as N → ∞, the difference between the original signal and the Fourier series representation should approach zero.

However, Gibbs (1839 – 1903) demonstrated the existence of overshoots at corners of pulses for example.



The Fourier series representation is off by approximately 9%

Dirichlet conditions: a periodic signal $x(t)$ has a Fourier series if it satisfies the following conditions:

1. x(t) is absolutely integrable over any period; that is

For any a:
$$\int_a^{a+T} |x(t)|\, dt < \infty$$

2. $x(t)$ has only a finite number of maxima and minima over any period

3. $x(t)$ has only a finite number of discontinuities over any period

Even and Odd functions:

1. $x(t)$ is an even function if: $x(-t)= x(t)$

2. $x(t)$ is an odd function if: $x(-t)= -x(t)$

- If $x(t)$ is even and $v(t)$ is even then $y(t)=x(t)\times v(t)$ is an even function
- If $x(t)$ is even and $v(t)$ is odd then $y(t)=x(t)\times v(t)$ is an odd function
- If $x(t)$ is odd and $v(t)$ is even then $y(t)=x(t)\times v(t)$ is an odd function
- If $x(t)$ is odd and $v(t)$ is odd then $y(t)=x(t)\times v(t)$ is an even function

Use of Symmetry:

1. If x(t) is an even function then

$$\int_{-a}^{a} x(t)dt = 2\int_{0}^{a} x(t)dt$$

2. If x(t) is an odd function then

$$\int_{-a}^{a} x(t)dt = 0$$

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \begin{cases} \frac{2}{T}\int_0^{\frac{T}{2}} x(t)dt & \text{when } x(t) \text{ is } \underline{\text{even}} \\ 0 & \text{when } x(t) \text{ is } \underline{\text{odd}} \end{cases}$$

$$a_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos(k\omega_0 t)dt = \begin{cases} \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos(k\omega_0 t)dt & \text{when } x(t) \text{ is } \underline{\text{even}} \\ 0 & \text{when } x(t) \text{ is } \underline{\text{odd}} \end{cases}$$

$$b_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin(k\omega_0 t)dt = \begin{cases} 0 & \text{when } x(t) \text{ is } \underline{\text{even}} \\ \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\sin(k\omega_0 t)dt & \text{when } x(t) \text{ is } \underline{\text{odd}} \end{cases}$$

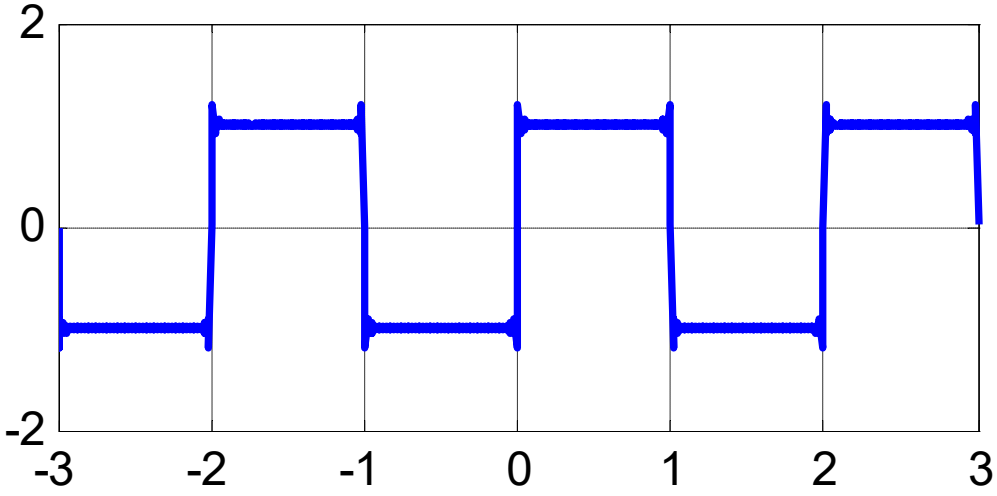
- Example 1: Find the Fourier series representation for $x(t)$ defined as

$x(t)=1$ $0< t <1$ and $x(t)=-1$ $-1< t <0$ and $T=2$ (period)

- Solution: $x(t)$ is an odd signal therefore:

$$a_0 = a_k = 0$$

$$b_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin(k\omega_0 t)dt = 2\int_0^1 \sin(k\pi t)dt = \frac{-2}{k\pi}\cos(k\pi t)\Big|_0^1 = \frac{-2}{k\pi}\{(-1)^k - 1\}$$



Example 2: find the Fourier series representation for $x(t)$ defined as
$x(t)=1$ for $-0.5< t <0.5$; and $x(t)=0$ for $-1< t <-0.5$ and $0.5< t <1$; and $T=2$ (period)

Solution: $x(t)$ is an even signal therefore: $b_k = 0$

$$a_0 = \frac{2}{2}\int_0^{0.5} 1dt = \frac{2}{2}(0.5) = 0.5$$

$$a_k = \frac{4}{2}\int_0^{0.5} \cos(k\pi t)dt = \frac{2}{k\pi}\sin(\frac{k\pi}{2})$$

Example 3: find the Fourier series representation for $x(t)$ defined as: T=2 (period), so $\omega_0 = 2\pi/T = \pi$

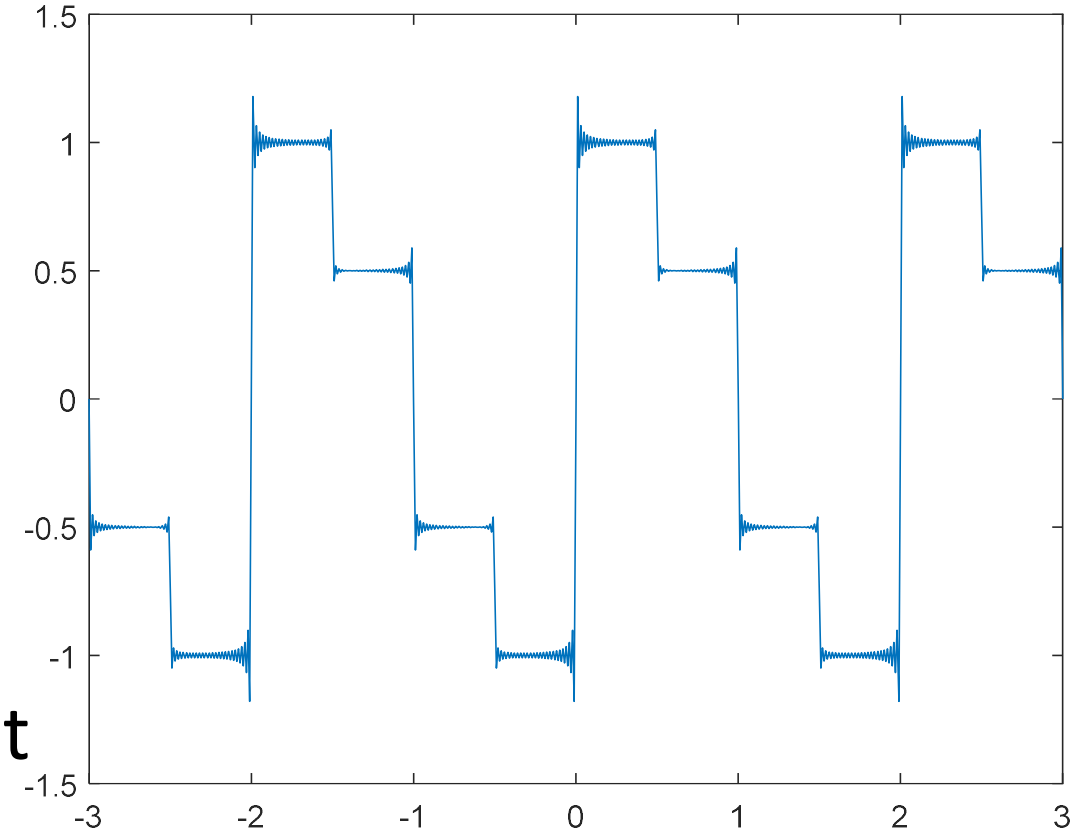
$$x(t) = \begin{cases} 1 & 0 < t < 0.5 \\ 0.5 & 0.5 < t < 1 \\ -1 & -0.5 < t < 0 \\ -0.5 & -1 < t < -0.5 \end{cases}$$

Solution: $x(t)$ is an odd signal therefore: $a_0 = a_k = 0$

$$b_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin(k\omega_0 t)dt = 2\int_0^1 x(t)\sin(k\pi t)dt = 2\left(\int_0^{0.5} 1\sin(k\pi t)dt + \int_{0.5}^1 (0.5)\sin(k\pi t)dt\right)$$

$$= 2\left(\frac{-1}{k\pi}\cos(k\pi t)\Big|_0^{0.5} + \frac{-1}{k\pi}\cos(k\pi t)\Big|_{0.5}^1\right) = \frac{1}{k\pi}\left(2 - \cos\left(\frac{k\pi}{2}\right) - \cos(k\pi)\right)$$

$$\Rightarrow x(t) = \sum_{k=1}^{N} [\frac{1}{k\pi}(2 - \cos(\frac{k\pi}{2}) - \cos(k\pi))]\sin(k\pi t)$$



MATLAB code

```
t = -3:0.01:3;
N = 100; w0 = pi;
x = 0; % No DC component
for k= 1:N
x = x + 1/(k*pi)*(2-cos(k*pi/2)-cos(k*pi))*sin(k*w0*t);
end
plot(t,x)
```

Complex Exponential Form:

The Fourier series of a periodic signal $x(t)$ can be expressed in the complex exponential form given by:

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)] = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad -\infty < t < \infty$$

$$c_k = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t) e^{jk\omega_0 t} dt \quad k = 0, \pm 1, \pm 2, \ldots$$

Proof:

Using Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k(\cos k\omega_0 t + j\sin k\omega_0 t) = c_0 + \sum_{k=1}^{\infty} c_k(\cos k\omega_0 t + j\sin k\omega_0 t)$$

$$+\sum_{k=-1}^{-\infty} c_k(\cos k\omega_0 t + j\sin k\omega_0 t) = c_0 + \sum_{k=1}^{\infty} c_k(\cos k\omega_0 t + j\sin k\omega_0 t) + \sum_{k=1}^{\infty} c_{-k}(\cos k\omega_0 t - j\sin k\omega_0 t)$$

$$= c_0 + \sum_{k=1}^{\infty}[(c_k + c_{-k})\cos k\omega_0 t + j(c_k - c_{-k})\sin k\omega_0 t]$$

Therefore:

$$a_0 = c_0; a_k = c_k + c_{-k}; b_k = j(c_k - c_{-k})$$

$$c_0 = a_0; c_k = \frac{a_k - jb_k}{2}; c_{-k} = \frac{a_k + jb_k}{2}$$

Example: find the complex Fourier series representation of a rectangular pulse train $x(t)$

$$c_k = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} x(t)e^{jk\omega_0 t} = \frac{1}{2}\int_{-1}^{1} x(t)e^{jk\pi t} = \frac{1}{2}\int_{-0.5}^{0.5} e^{jk\pi t} = \frac{1}{2jk\pi}(e^{j\frac{k\pi}{2}} - e^{-j\frac{k\pi}{2}})$$

$$= \frac{1}{k\pi}\sin(\frac{k\pi}{2}) \qquad k = 0, \pm 1, \pm 2, \ldots$$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{k\pi}\sin(\frac{k\pi}{2})e^{jk\pi t}$$

Finding a_0, a_k and b_k from c_k:

$$a_0 = c_0 = \frac{1}{2}$$

$$a_k = c_k + c_{-k} = \frac{1}{k\pi}\sin(\frac{k\pi}{2}) + \frac{1}{-k\pi}\sin(\frac{-k\pi}{2}) = \frac{2}{k\pi}\sin(\frac{k\pi}{2})$$

$$b_k = j(c_k - c_{-k}) = j(\frac{1}{k\pi}\sin(\frac{k\pi}{2}) - \frac{1}{-k\pi}\sin(\frac{-k\pi}{2})) = 0$$

$$x(t) = \frac{1}{2} + \sum_{k=1}^{N}[\frac{2}{k\pi}\sin(\frac{k\pi}{2})\cos(k\pi t)]$$

MATLAB code

```
t = -3:0.01:3;
N = 500;a0 = 0.5;w0 = pi;j=sqrt(-1);
x = a0*ones(1, length(t));   %  dc component
for k= -N:N
    if k~=0
       x = x + (1/(k*pi))*sin(k*pi/2)*exp(j*k*w0*t);
    end
end
plot(t,x,'linewidth',2)
```

Parseval's theorem: assume that $x(t)$ is a periodic signal with period T. The average power P of the signal is defined by:

$$P = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} x^2(t)dt$$

By Parseval's theorem, the average power P of the signal $x(t)$ is given by:

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Proof:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad x(t)^* = \sum_{n=-\infty}^{\infty} c_n e^{-jn\omega_0 t}$$

$$P = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} (c_k e^{jk\omega_0 t})(c_n e^{jn\omega_0 t})^* dt = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} (c_k c_n^*)(e^{j(k-n)\omega_0 t})dt$$

$$P = \begin{cases} \sum_{k=-\infty}^{\infty} |c_k|^2 & k = n \\ 0 & k \neq n \end{cases}$$