EE 362: Signals and Systems

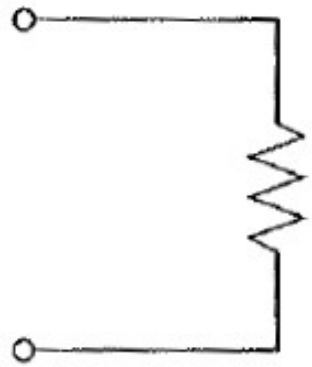
Time-Domain Models of Systems

Lecture 5

Differential Equation Models

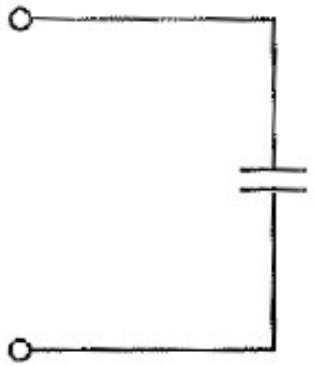
Continuous-time systems are often specified by an input/output differential equation that can be generated by application of the laws of physics.

Electrical Circuits:



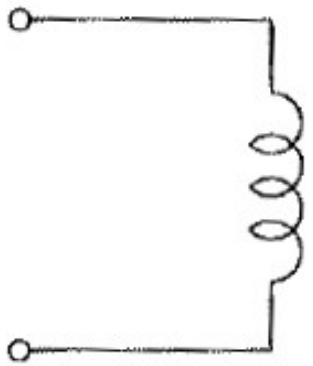
Resistor:

$v(t)=R\,i(t)$



Capacitor:

$$i(t) = C\frac{dv(t)}{dt}$$

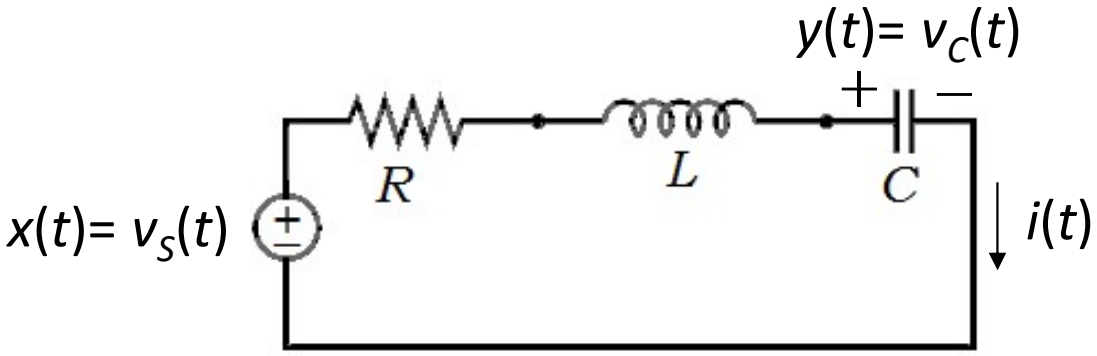


Inductor:

$$v(t) = L\frac{di(t)}{dt}$$

Example: Series *RLC*

Find the input/output differential equation if the input is $x(t)=v_S(t)$ and the output is the voltage across the capacitor

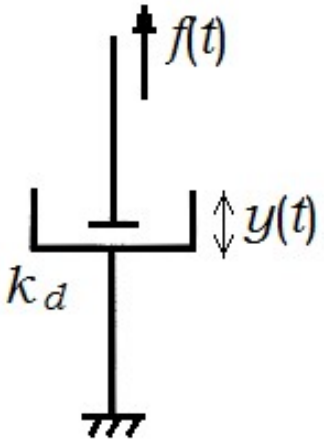


$$v_S(t) = Ri(t) + L\frac{di(t)}{dt} + v_C(t)$$

From KVL

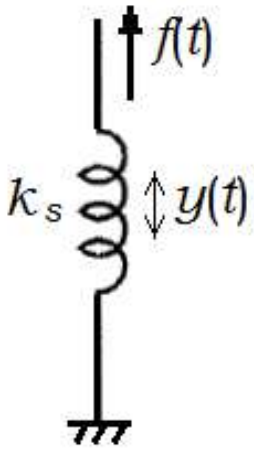
$$i(t) = C\frac{dv_C(t)}{dt} \longrightarrow LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

Mechanical Systems:



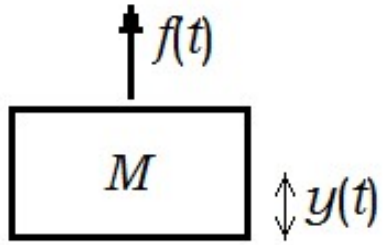
Dashpot:

$$f(t) = k_d \frac{dy(t)}{dt}$$



Spring:

$$f(t) = k_s y(t)$$



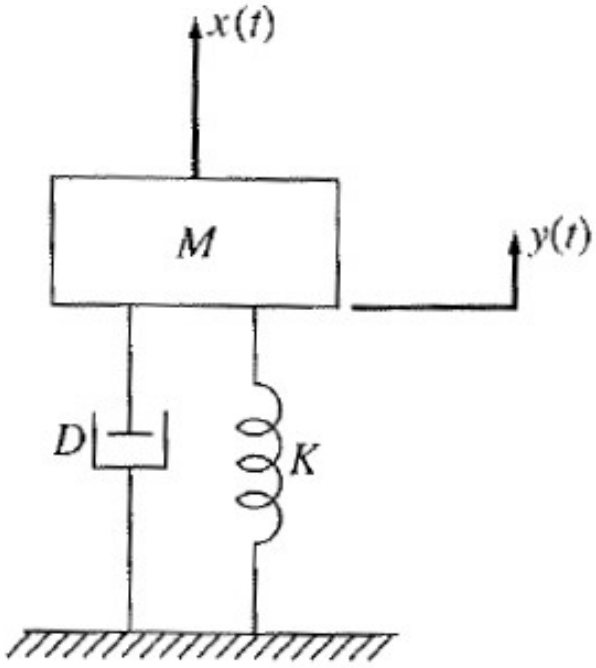
Mass:

$$f(t) = M \frac{d^2 y(t)}{dt^2}$$

$f(t)$: Force
$y(t)$: Position
K_d: Damping constant
K_s: Stiffness of spring
M: Mass

Note: If we replace $dy(t)/dt$ with $v(t)$ (speed) then: Dashpot acts like a resistor, Spring acts like an inductor and Mass acts like a capacitor

Example: Mass-Spring-Damper System
A second order differential equation

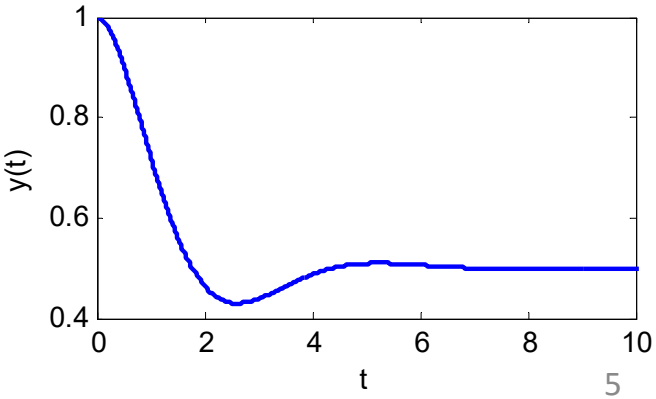


$$M\frac{d^2y(t)}{dt^2}+D\frac{dy(t)}{dt}+Ky(t)=x(t)$$

Example: Find the response of a Mass-Spring-Damper System to $x(t)$=1, D=1.5, K=2, M=1, $y(0)$=1, dy(0)/dt=0

MATLAB code:

```
x=1;
y=dsolve('D2y=-1.5*Dy-2*y+x','Dy(0)=0',
'y(0)=1')
```



Solution of Differential Equations

- Several Methods:
  - Solving for the homogeneous solution and the nonhomogeneous solution separately (covered in previous courses).
  - Laplace Transform (will be covered in Ch. 6)
  - Numerical Methods:
    - Discretization (numerical method)
    - MATLAB using ODE solvers (e.g., Runge-Kutta which is accessible with command ode45)

Discretization

- A differential equation can be discretized in time, resulting in a difference equation that can then be solved by **recursion**. This yields an approximate discrete-time representation of the continuous-time system defined by the given input/output differential equation.
- Consider the following LTI continuous-time system

$$\frac{dy(t)}{dt} = -ay(t) + bx(t)$$

- where a and b are constants.

- This system can be discretized in time by setting $t = nT$, where T is a fixed positive number and n takes on integer values only.

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = -ay(nT) + bx(nT)$$

- The derivative can be approximated as:

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT + T) - y(nT)}{T}$$

- If T is suitably small, the approximation will be accurate. This approximation is called Euler approximation.

- Then

$$\frac{y(nT + T) - y(nT)}{T} = -ay(nT) + bx(nT)$$

- To be consistent with the notation of the discrete-time signals, let

$$x[n] = x(t)|_{t=nT} \text{ and } y[n] = y(t)|_{t=nT}$$

- Then

$$\frac{y[n + 1] - y[n]}{T} = -ay[n] + bx[n]$$

- Multiply both sides by T and replace n by $n-1$

$$y[n] - y[n-1] = -aTy[n-1] + bTx[n-1]$$

OR

$$y[n] = (1 - aT)y[n-1] + bTx[n-1]$$

The solution of the above equation with initial condition $y[0]$ and with $x[n] = 0$ for all n is given by

Approximate solution

$$y[n] = (1 - aT)^n y[0], \quad n = 0, 1, 2, \ldots$$

Approximate Solution vs Exact Solution

- Using Laplace transform, the exact solution with initial condition $y(0)$ and with zero input is given by

$$y(t) = e^{-at}y(0), t \geq 0$$

- To compare, set $t = nT$:

$$y[n] = e^{-anT}y[0], \quad n = 0, 1, 2, \ldots$$

- Then

$$y[n] = (e^{-aT})^n y[0], \quad n = 0, 1, 2, \ldots$$

- But

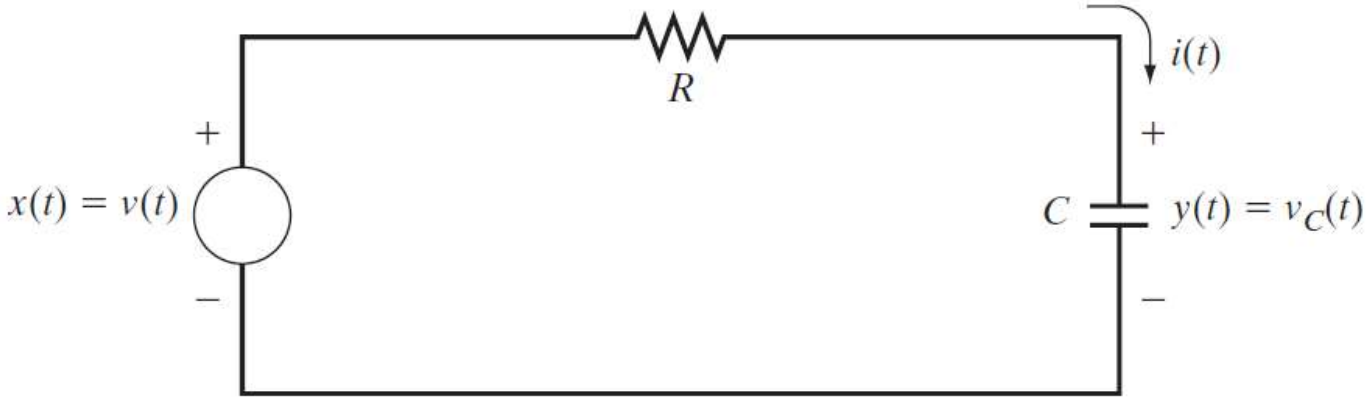
$$e^{-aT} = 1 - aT + \frac{a^2T^2}{2} - \frac{a^3T^3}{6} + \cdots$$

- Then,

$$y[n] = \left(1 - aT + \frac{a^2T^2}{2} - \frac{a^3T^3}{6} + \cdots\right)^n y[0], \quad n = 0, 1, 2, \ldots$$

- Then the approximation will be accurate if $1 - aT$ is a good approximation to e^{-aT}
- This will be the case if the magnitude of aT is much smaller than 1.

- Example: Consider the following RC circuit;



$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t) \qquad a = 1/RC \qquad b = 1/RC$$

- Using:

$$y[n] = (1 - aT)y[n-1] + bTx[n-1]$$

- We get:

$$y[n] = \left(1 - \frac{T}{RC}\right)y[n-1] + \frac{T}{RC}x[n-1]$$

- Applying the initial conditions, we can solve for the voltage across the capacitor recursively.
- Then

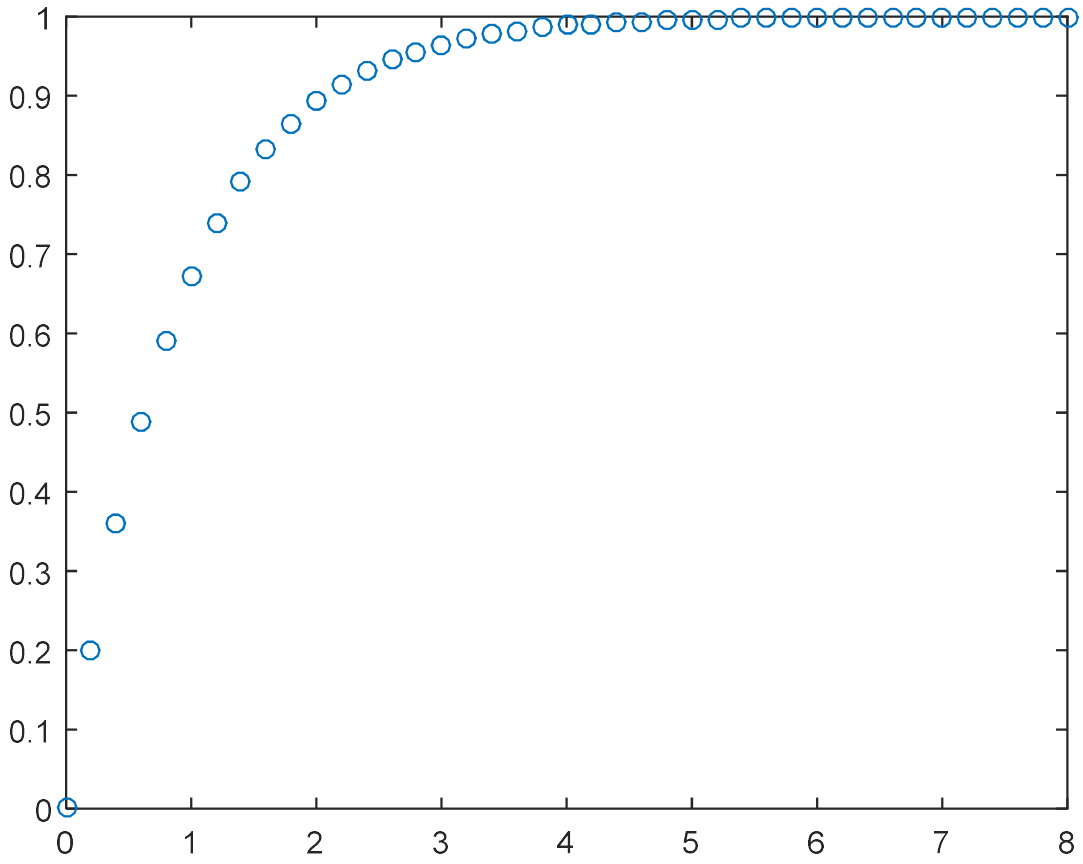
$$y[n] = -\sum_{i=1}^{N} a_i y[n-i] + \sum_{i=0}^{M} b_i x[n-i]$$

$$y[n] = \left(1 - \frac{T}{RC}\right) y[n-1] + \frac{T}{RC} x[n-1]$$

- $a_1 = -(1 - T/RC), b_0 = 0, b_1 = T/RC$

- Using MATLAB: Assume $R = C = 1, y[0] = 0,$ $x(t) = 1$ for $t \geq 0$, and $T = 0.2$

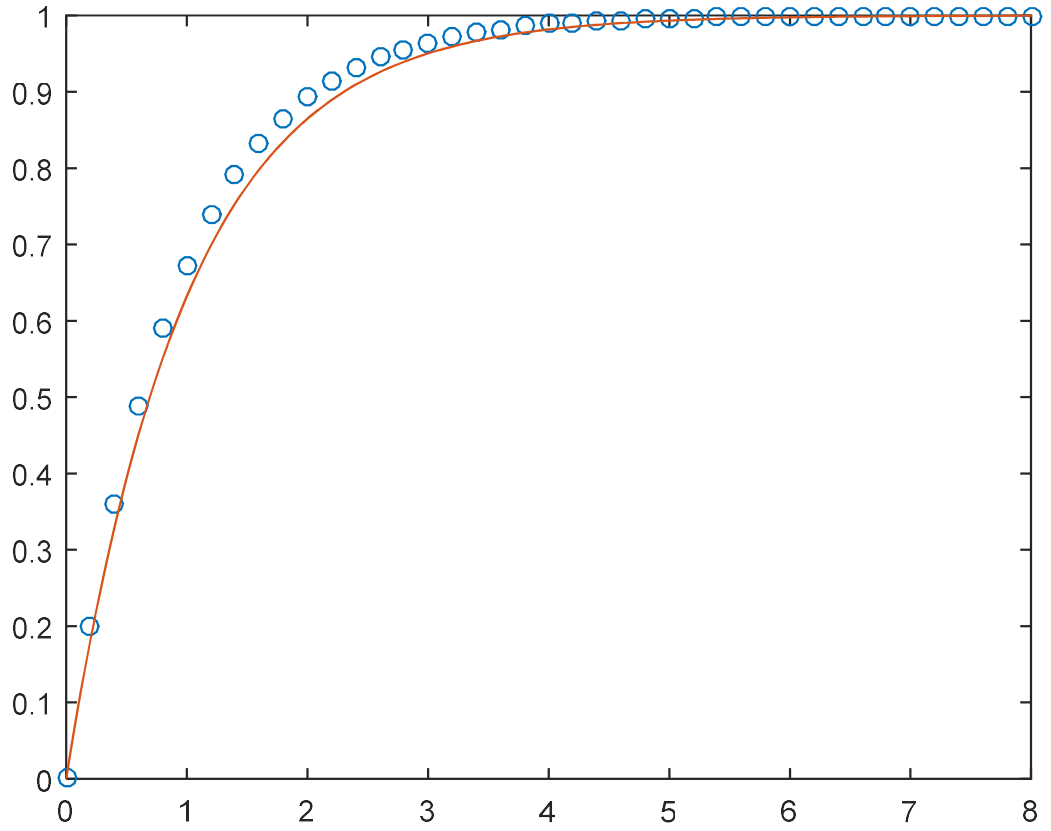
```
R = 1; C = 1; T = 0.2;
a = -(1-T/R/C); b = T/R/C;
y0=0; x0=1;
n=1:8/T; % you can choose any number for x-axis
x=ones(1,length(n));
y1=ones(1,length(n));
for i=1:length(n)
    y1(i)=-a*y0+b*x0;
    y0=y1(i);
    x0=x(i);
end
y1=[0 y1];
n=0:8/T;
plot(n*T,y1,'o')
```



- Comparing with exact solution

```
R = 1; C = 1; T = 0.2;
a = -(1-T/R/C); b = T/R/C;
y0=0; x0=1;
n=1:8/T; % you can choose any number for x-axis
x=ones(1,length(n));
y1=ones(1,length(n));
for i=1:length(n)
    y1(i)=-a*y0+b*x0;
    y0=y1(i);
    x0=x(i);
end
y1=[0 y1];
n=0:8/T;
plot(n*T,y1,'o')

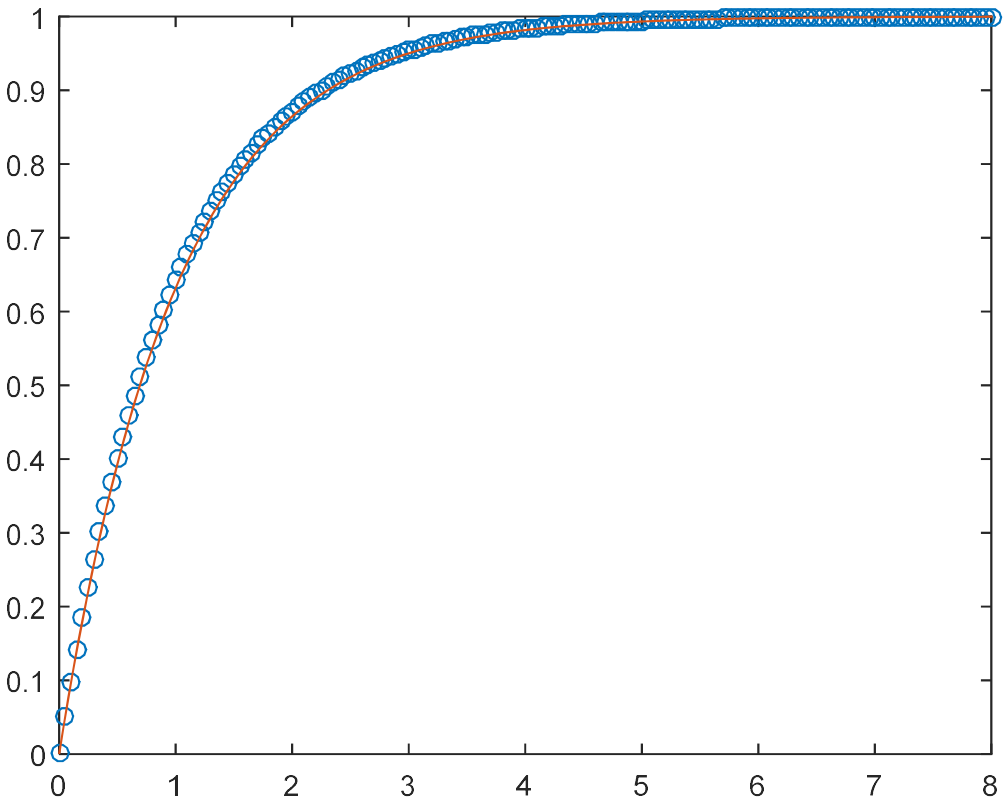
% Exact solution
t = 0:0.04:8;
y2= 1 - exp(-t);
plot(n*T,y1,'o',t,y2,'-')
```



- Decrease T:

```
R = 1; C = 1; T = 0.05;
a = -(1-T/R/C); b = T/R/C;
y0=0; x0=1;
n=1:8/T; % you can choose any number for x-axis
x=ones(1,length(n));
y1=ones(1,length(n));
for i=1:length(n)
    y1(i)=-a*y0+b*x0;
    y0=y1(i);
    x0=x(i);
end
y1=[0 y1];
n=0:8/T;
plot(n*T,y1,'o')

% Exact solution
t = 0:0.04:8;
y2= 1 - exp(-t);
plot(n*T,y1,'o',t,y2,'-')
```



- The discretization technique for the first-order differential equations can be generalized to second- and higher-order differential equations.

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT+T) - y(nT)}{T}$$

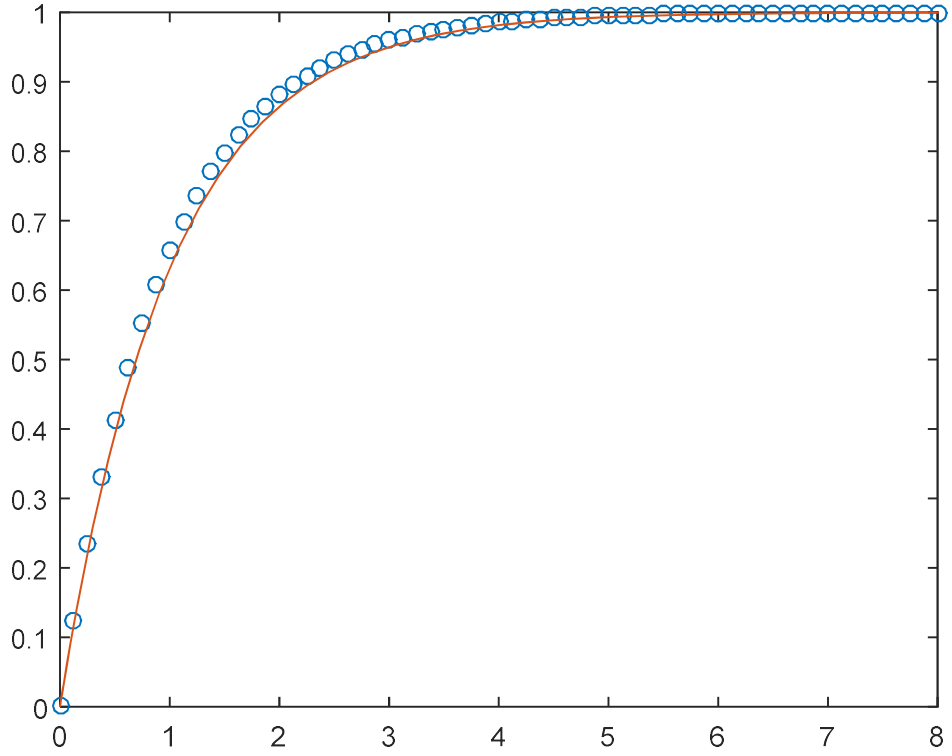
$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \frac{dy(t)/dt|_{t=nT+T} - dy(t)/dt|_{t=nT}}{T}$$

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \frac{y(nT+2T) - 2y(nT+T) + y(nT)}{T^2}$$

Using MATLAB ODE Solver

- Consider $\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$

```
tspan = [0 8];
t,y3]=ode45(@ex_ode_fun,tspan,y0);
n=length(t)-1;
T=8/n;
y0=0;
R=1; C=1; x0=1;
a = -(1-T/R/C); b = T/R/C;
x=ones(1,length(t)-1);
y1=ones(1,length(t)-1);
for i=1:length(t)-1
    y1(i)=-a*y0+b*x0;
    y0=y1(i);
    x0=x(i);
end
y1=[0 y1];
n=0:8/T;
plot(n*T,y1,'o',t,y3,'-')
```




```
function dy = ex_ode_fun(t,y);
R=1; C=1; x=1;
dy=x/R/C-y/R/C;
```

Using MATLAB Symbolic Math Solution

- For the same example

```
R=1; C=1; x0=1;
%y=dsolve('Dy=1/R/C*x0-
1/R/C*y','y(0)=0')
y=dsolve('Dy=1-y','y(0)=0')
```

```
y =

1 - exp(-t)
```

- Example:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y(t) = 0; \quad \left.\frac{dy}{dt}\right|_{t=0} = 0, \quad y(0) = 1$$

```
y=dsolve('D2y=-4*Dy-13*y','Dy(0)=0','y(0)=1')
```

y =

cos(3*t)*exp(-2*t) + (2*sin(3*t)*exp(-2*t))/3

Convolution of continuous-time systems

Definition: the convolution of two continuous-time signals $x(t)$ and $v(t)$ is defined as:

$$x(t) * v(t) = \int_{-\infty}^{\infty} x(t)v(t-\lambda)d\lambda$$

RHS is called the "*convolution integral*"

Property: the convolution operation $x(t)*v(t)$ is "commutative": $x(t)*v(t) = v(t)*x(t)$

$$x(t) * v(t) = \int_{\lambda=-\infty}^{\lambda=\infty} x(t)v(t-\lambda)d\lambda = \int_{\lambda=-\infty}^{\lambda=\infty} v(t)x(t-\lambda)d\lambda = v(t) * x(t)$$

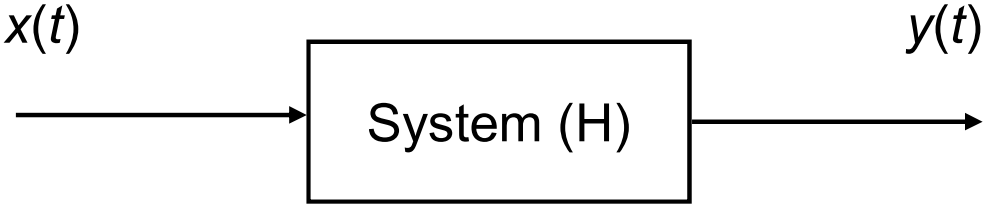
If $x(t)$ and $v(t)$ are zero for all $t < 0$, then $x(t) = 0$ for all $t < 0$ and $v(\lambda - t) = 0$ for all $\lambda - t < 0$ (or $\lambda < t$), therefore:

$$x(t) * v(t) = \begin{cases} \int_{\lambda=0}^{\lambda=t} x(t)v(t-\lambda)d\lambda = \int_{\lambda=0}^{\lambda=t} v(t)x(t-\lambda)d\lambda & t \geq 0 \\ 0 & t < 0 \end{cases}$$

If a linear time-invariant continuous-time system is casual ($h(t) = 0$ for $t < 0$) and also assume that $x(t)$ is zero for all $t < 0$, then the output $y(t)$ is given by:

$$y(t) = x(t) * h(t) = \int_{\lambda=0}^{\lambda=t} x(\lambda)\ h(t-\lambda)\ d\lambda \xrightarrow{by\ commutitively} = \int_{\lambda=0}^{\lambda=t} h(\lambda)x(t-\lambda)d\lambda \qquad t \geq 0$$

where $h(t)$ is the impulse response of the system {$h(t)$ is the output of the system when the input $x(t)=\delta(t)$}



Example: find the convolution of $x(t)=e^{t}u(t)$ and $v(t)=e^{-t}u(t)$, where a and b are nonzero constant numbers and $u(t)$ is the unit step function:

$$x(t)*h(t)=\int_{\lambda=0}^{\lambda=t}x(\lambda)h(t-\lambda)d\lambda=\int_{0}^{t}e^{\lambda}u(\lambda)\;e^{-(t-\lambda)}u(t-\lambda)\;d\lambda=\int_{0}^{t}e^{\lambda}e^{-t+\lambda}d\lambda$$

$$=e^{-t}\int_{0}^{t}e^{2\lambda}d\lambda=\frac{1}{2}(e^{t}-e^{-t})$$

Graphical Approach to Convolution: to compute $x(t)*h(t)$, it is useful to graph the functions specially when $x(t)$ *and* $h(t)$ are piecewise

$$x(t) * h(t) = \int_{\lambda=0}^{\lambda=t} x(\lambda)\ h(t-\lambda)\ d\lambda$$

Step 1: Graph $x(\lambda)$ and $h(-\lambda)$ as functions of λ. The function $h(-\lambda)$ is equal to the function $h(\lambda)$ reflected about the vertical axis.

Step 2: Graph $h(t - \lambda)$ for an arbitrary value of t such that $t < 0$

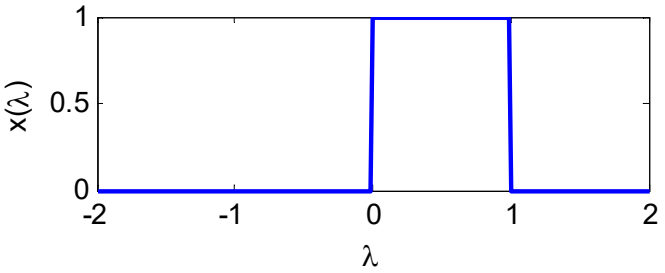
Step 3: Change t and slide $h(t - \lambda)$ to the right until there is a nonzero overlap between $h(t - \lambda)$ and $x(\lambda)$.

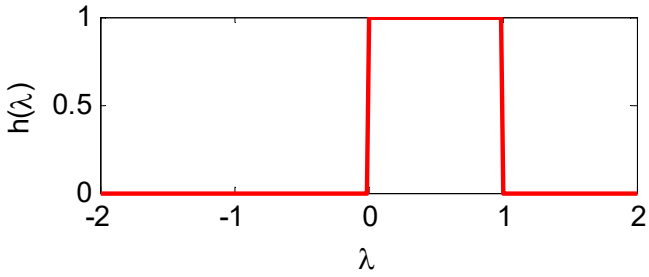
Step 4: Continue sliding $h(t - \lambda)$ to the right and determine the interval of t for which the product $x(\lambda)h(t - \lambda)$ has the same analytical form. Integrate the product $x(\lambda)h(t - \lambda)$ as a function of λ with the limits.

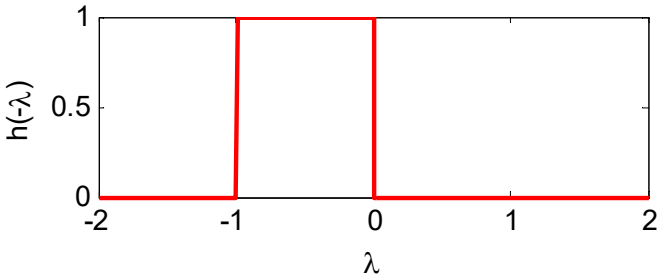
Step 5: Repeat Step 4 until $x(t)*h(t)$ is computed for all $t > 0$

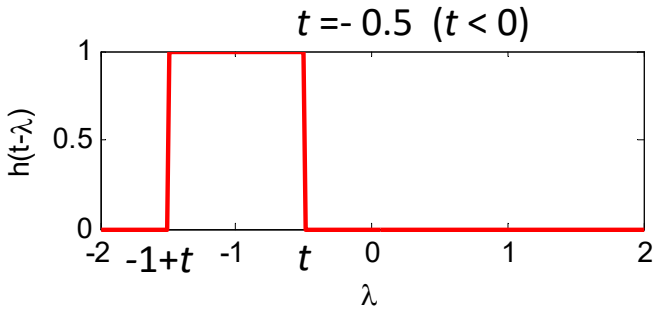
Example: find the convolution of $x(t)=u(t)-u(t-1)$ and $h(t)= u(t)-u(t-1)$:

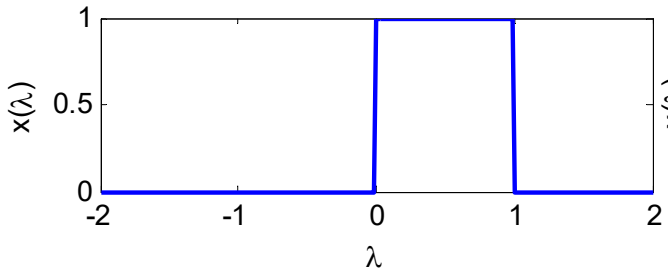
$$x(t) * h(t) = \int_{\lambda=0}^{\lambda=t} x(\lambda)\, h(t-\lambda)\, d\lambda = \begin{cases} 0 & t < 0 \\ \int_0^t 1\, d\lambda = t & 0 \le t < 1 \\ \int_{-1+t}^{1} 1\, d\lambda = 2-t & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

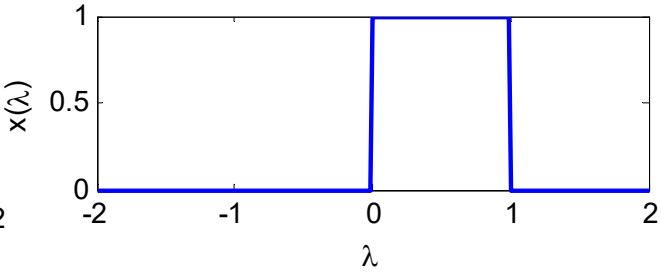

x(λ)
1
0.5
0
-2
-1
0
1
2
λ

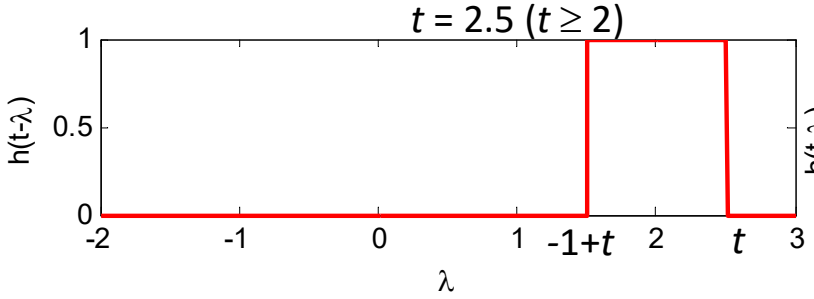

h(λ)
1
0.5
0
-2
-1
0
1
2
λ

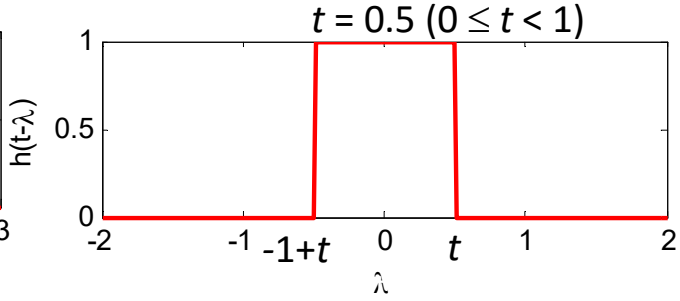

h(-λ)
1
0.5
0
-2
-1
0
1
2
λ

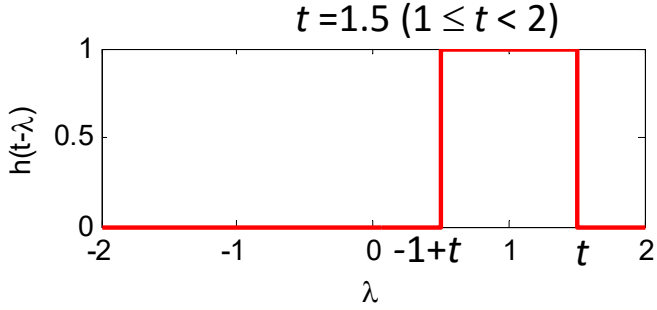

t =- 0.5 (t < 0)
h(t-λ)
1
0.5
0
-2
-1+t
-1
t
0
1
2
λ


x(λ)
1
0.5
0
-2
-1
0
1
2
λ


x(λ)
1
0.5
0
-2
-1
0
1
2
λ

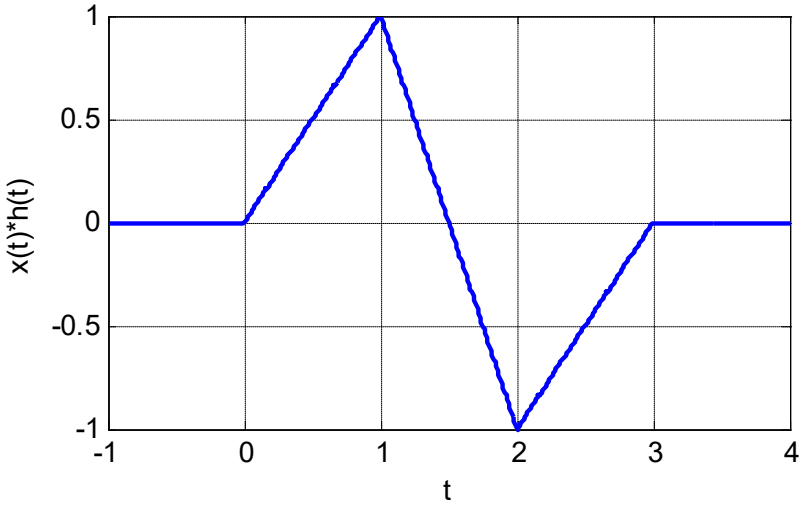

t = 2.5 (t ≥ 2)
h(t-λ)
1
0.5
0
-2
-1
0
1
-1+t
2
t
3
λ


t = 0.5 (0 ≤ t < 1)
h(t-λ)
1
0.5
0
-2
-1
-1+t
0
t
1
2
λ


t =1.5 (1 ≤ t < 2)
h(t-λ)
1
0.5
0
-2
-1
0
-1+t
1
t
2
λ

Example: find the convolution of $x(t)=u(t)-2u(t-1)+u(t-2)$ and $h(t)= u(t)-u(t-1)$ for all t:

$$x(t)*h(t)=\int_0^t x(\lambda)\,h(t-\lambda)\,d\lambda=\begin{cases}0 & t\le 0\\ \int_0^t (1)(1)\,d\lambda = t & 0\le t<1\\ \int_{-1+t}^1 (1)(1)\,d\lambda+\int_1^t (1)(-1)\,d\lambda = 3-2t & 1\le t<2\\ \int_{-1+t}^2 (1)(-1)\,d\lambda = -3+t & 2\le t<3\\ 0 & t\ge 2\end{cases}$$

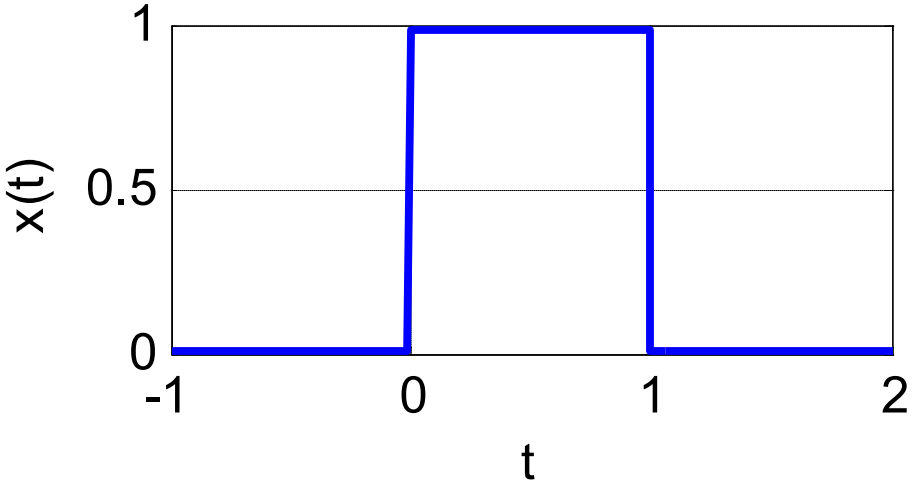


Example: find the convolution of $x(t)=e^{-t}u(t)$ and $v(t)= u(t)-u(t-1)$ for all t:

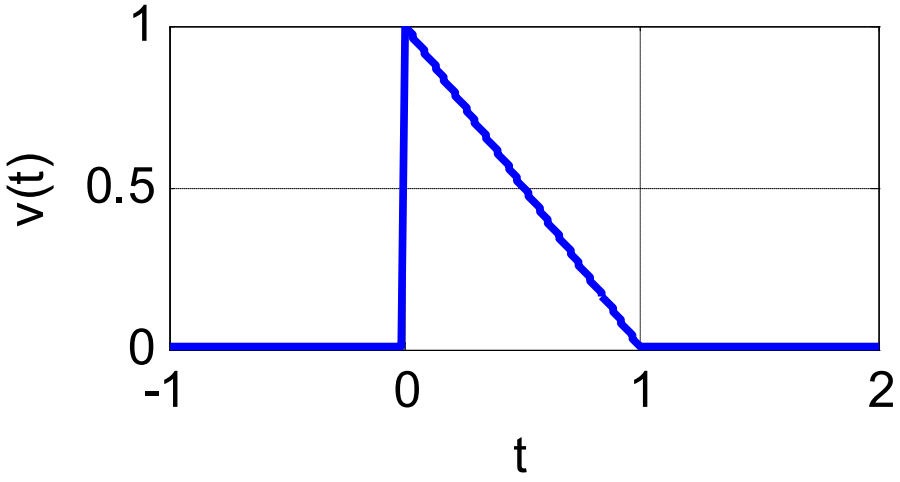
$$x(t)*v(t)=\int_0^t x(\lambda)\,v(t-\lambda)\,d\lambda=\begin{cases}0 & t<0\\ \int_0^t e^{-\lambda}d\lambda=1-e^{-t} & 0\le t<1\\ \int_{-1+t}^{t} e^{-\lambda}d\lambda=e^{-t+1}-e^{-t} & t\ge 1\end{cases}$$

Example: find the convolution of $x(t)$ and $v(t)$ for all t:

$$x(t)=u(t)-u(t-1)$$



$$v(t)=(1-t)(u(t)-u(t-1))$$



$$x(t) * v(t) = \int_0^t x(\lambda)\ v(t-\lambda)\ d\lambda = \begin{cases} 0 & t < 0 \\ \int_0^t (1)(1-t+\lambda)\ d\lambda = t - \frac{t^2}{2} & 0 \le t < 1 \\ \int_{-1+t}^1 (1)(1-t+\lambda)\ d\lambda = \frac{t^2}{2} - 2t + 2 & t \ge 1 \end{cases}$$