EE 362: Signals and Systems

Time-Domain Models of Systems

Lecture 3

Time-Domain Models of Systems

- Input/Output Representation of Discrete-Time Systems
- Convolution of Discrete-Time Signals
- Difference Equation Models
- Differential Equation Models
- Solution of Differential Equations
- Convolution Representation of Continuous-Time Systems

Input/Output Representation of Discrete-Time System

Consider the N-point moving average (MA) filter.

$$y[n] = \frac{1}{N}\left[x[n] + x[n-1] + x[n-2] + \cdots + x[n-N+1]\right]$$

OR

$$y[n] = \sum_{i=0}^{N-1} \frac{1}{N} x[n-i]$$

where N is a positive integer.

In general, $y[n] = \sum_{i=0}^{N-1} \frac{1}{N} x[n-i]$ can be expressed as follows:

$$y[n] = \sum_{i=0}^{N-1} \omega_i x[n-i]$$

which represents a **causal** **linear** **time-invariant** discrete-time system. Here ω_i (i.e. $\omega_0, \omega_1, \omega_2, \ldots, \omega_{N-1}$) are real numbers and called the weights of the linear combination of the above system.

- N-Point Exponentially Weighted Moving Average (EWMA) Filter

$$y[n] = \sum_{i=0}^{N-1} a\left(b^i x[n-i]\right)$$

where b is a real number with $0 < b < 1$ and a is a positive constant given by

$$a = \frac{1-b}{1-b^N} \quad and \quad \omega_i = ab^i$$

If $b = 0$, then $a = 1 \rightarrow$ no filtering

If $b = 1$, then $a = \frac{1}{N} \rightarrow$ MA filter

Example: Determine the input/output relationship of a 3-point EWMA filter with $b = 0.5$

$$a = \frac{1 - 0.5}{1 - 0.5^3} = 0.571$$

$$\omega_0 = a = 0.571, \omega_1 = ab = 0.286, and$$

$$\omega_2 = ab^2 = 0.143$$

Then

$$y[n] = 0.571x[n] + 0.286x[n-1] + 0.143x[n-2]$$

If $b = 1$,

$$y[n] = 0.333x[n] + 0.333x[n-1] + 0.333x[n-2]$$

A General Class of Systems

$$y[n] = \sum_{i=0}^{n} \omega_i x[n-i], \qquad n \geq 0$$

Therefore, causal linear time-invariant discrete-time systems with input $x[n] = 0 \;\; \forall\, n < 0$ can be represented by the above expression.

This type of systems is usually expressed in terms of its unit-pulse response

Unit-Pulse Response

The unit-pulse response, denoted by $h[n]$, is the output response of the system resulting from the application of the unit-pulse $\delta[n]$. Recall that $\delta[0] = 1$ and $\delta[n] = 0 \; \forall \; n \neq 0$

Since $\delta[n] = 0$ for $n < 0$ and by causality the unit-pulse response $h[n]$ must be zero for all integers $n < 0$.

To compute the unit-pulse response, simply insert $x[n] = \delta[n]$

$$h[n] = \sum_{i=0}^{n} \omega_i \delta[n-i], \qquad n \geq 0$$

When $i = n$, $\delta[n-i] = 1$ and $\delta[n-i] = 0 \; \forall \; i \neq n$

Then,

$$h[n] = \omega_n, \qquad n \geq 0$$

Therefore,

$$y[n] = \sum_{i=0}^{n} h[i]x[n-i], \qquad n \geq 0$$

Convolution

The expression $\sum_{i=0}^{n} h[i]x[n-i]$ is called *convolution* of $h[n]$ and $x[n]$ and is denoted by the symbol "$*$"

$$h[n] * x[n] = \sum_{i=0}^{n} h[i]x[n-i]$$

Then,

$$y[n] = h[n] * x[n], \qquad n \geq 0$$

This convolution representation of the system is a time-domain model.

Convolution of Discrete-Time Signals

Given two discrete-time signals, $x[n]$ and $v[n]$, the convolution of these two signals is:

$$x[n] * v[n] = \sum_{i=-\infty}^{\infty} x[i]v[n-i]$$

The convolution operation $x[n] * v[n]$ is commutative; i.e., $x[n] * v[n] = v[n] * x[n]$

$$\sum_{i=-\infty}^{\infty} x[i]v[n-i] = \sum_{i=-\infty}^{\infty} v[i]x[n-i]$$

If $x[n]$ and $v[n]$ are zero for all integers $n < 0$, then $x[i] = 0$ for all integers $i < 0$ and $v[n-i] = 0$ for all integers $n < i$. Thus,

$$x[n] * v[n] = \begin{cases} 0, & n = -1, -2, \ldots \\ \sum_{i=0}^{n} x[i]v[n-i] & n = 0, 1, 2, \ldots \end{cases}$$

OR

$$x[n] * v[n] = \begin{cases} 0, & n = -1, -2, \ldots \\ \sum_{i=0}^{n} v[i]x[n-i] & n = 0, 1, 2, \ldots \end{cases}$$