DTFT

Note Title 5/1/2018

DTFT $\longrightarrow$ $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$

e^{-n} $\qquad$ $e^{-\alpha}$

Z-Transform $\longrightarrow$ $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, \rho^{-n} e^{-j\Omega n}$

$= \sum_{n=-\infty}^{\infty} x[n] \left(\rho e^{j\Omega}\right)^{-n}$ $\qquad$ If we define $z = \rho e^{j\Omega}$

$\therefore$ $\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}}$ $\leftarrow$ Z-Transform

Ex 1

$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{o.w.} \end{cases}$

$X(z) = \sum_{n=-\infty}^{\infty} \delta[n]\, z^{-n} = \delta[0]\, z^{0} = 1$

Ex 2 $\qquad u[n]$

$X(z) = \sum_{n=-\infty}^{\infty} u[n]\, z^{-n} = \sum_{n=0}^{\infty} z^{-n}$

$\therefore X(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \cdots$ $\qquad * z^{-1}$

$(z^{-1})X(z) = (z^{-1})(1 + z^{-1} + z^{-2} + z^{-3} + \cdots)$

$= (z + 1 + z^{-1} + z^{-2} + \cdots) - (\cdots + z^{-2} + z^{-1} + 1)$

$= z$

$z = \rho e^{j\Omega}$

$\therefore \boxed{X(z) = \frac{z}{z-1}}$ $\quad z \neq 1$ or $\rho \neq \frac{1}{e^{j\Omega}}$

$u[n] \longleftrightarrow \frac{z}{z-1}$

Ex 3

$x[n] = \delta[n-q]$

$X(z) = \sum_{n=-\infty}^{\infty} \delta[n-q]\, z^{-n}$; $\delta[n-q] = 1$ if $n=q$

$\therefore X(z) = z^{-q} = \frac{1}{z^q}$

$\therefore \quad \delta[n-4] \longleftrightarrow \frac{1}{z^4}$ or z^{-4}

Ex 4

$x[n] = a^n u[n]$

$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} a^n u[n] z^{-n}$

$= 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$

$(z-a) X(z) = (z-a)\left(1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots\right)$

$= \left(z + a + a^2 z^{-1} + a^3 z^{-2}\right) - \left(a + a^2 z^{-1} + a^3 z^{-2} + a^4 z^{-3} + \cdots\right)$

$= z$

$\therefore X(z) = \frac{z}{z-a}$ or $\boxed{a^n u[n] \longleftrightarrow \frac{z}{z-a}}$ $\quad |z| > a$

$$x[n] = \delta[n] + 2\delta[n-1]$$

$$\delta[n] \leftrightarrow 1$$

$$\delta[n-1] \leftrightarrow z^{-1} = \frac{1}{z}$$

$$\therefore X(z) = 1 + \frac{2}{z} = \frac{z+2}{z}$$

$$x[n] = u[n-2] \quad \leftrightarrow \quad \frac{z}{z-1} z^{-2}$$

$$a^n u[n] \leftrightarrow \frac{z}{z-a}$$

$$x[n-k]\, u[n-k] \leftrightarrow X(z)\, z^{-k}$$

$$a^n = a^{n-1} a$$

$$a\left(a^{n-1} u[n-1]\right)$$

Ex $\quad x[n] = a^n u[n-1]$

$X(z) = ?$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=1}^{\infty} a^n z^{-n} = a z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$$

$$(z-a) X(z) = \left(a + a^2 z^{-1} + a^3 z^{-2} + \cdots\right) - \left(a^2 z^{-1} + a^3 z^{-2} + a^4 z^{-3} + \cdots\right)$$

$$= a$$

$$X(z) = \frac{a}{z-a}$$